An Informal Thesis Proposal

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An *S space* is a regular Hausdorff hereditarily separable topological space that is not hereditarily Lindelöf.

An *L space* is a regular Hausdorff hereditarily Lindelöf topological space that is not hereditarily separable.

MATHEMATICS COLLOQUIUM

WHAT'S HAPPENING IN THE MATHEMATICAL SCIENCES?

DEPARTMENT OF MATHEMATICS
UNIVERSITY OF NEBRASKA AT OMAHA

WHEN: Monday, April 24, 2000 at 2:30 pm
WHERE: Durham Science Center, Room 255
WHAT:

Judy Roitman

of the Department of Mathematics, University of Kansas, Lawrence, will give a talk on

A Survey of the S and L Space Problem

ABSTRACT:
The problem of S and L spaces is one of the two paradigm problems of set theoretic topology (the other being the normal Moore space problem). It is a paradigm problem because

(a) it grows out of very basic questions about characterizations of common mathematical objects, in this case the real numbers;
(b) it has led to important developments in set theory;
(c) it has deep connections to other problems in both set theory and topology.

In addition, it has the virtue that one of the most important questions associated with it --- is there an L space? --- remains open. This talk will be a survey of some of the developments in the area of S and L spaces, as a way of introducing set theory and set theoretic topology to a general audience.

Refreshments served 30 minutes prior to the talk in DSC 255

"So the big questions remaining are: Is

'there are no L-spaces'

consistent. . . ? Handbook of Set Theoretical Topology (1984)

"No example of an L-space is known within the realm of classical set theory. . . . " General Topology (1989)

"At the time of this writing it is still an open problem whether ZFC proves the existence of an L-space." Discovering Modern Set Theory II (1997)

"It is still unknown whether

'there are no L-spaces'

is consistent (i.e., cannot be disproved by our usual axioms of ZFC) or not." Encyclopedia of General Topology (2003)

Topology News

11 Dec 2004

Topology News Index

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Date: 11 Dec 2004
From: Topology News
Subject: Topology News, December 2004

Topology News, December 2004

Knots in Washington XX
Solovyov conference
BEST 2005
Workshop in Geometric Topology
CAT'05
CITA-2005

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L-spaces exist

Justin Moore has announced a solution to the L-space problem in
set-theoretic topology by constructing, without additional set-theoretic
assumptions, a hereditarily Lindelof regular topological space which is
not separable.
A recent preprint is available from the author's homepage:
http://diamond.boisestate.edu/~justin/preprints.html

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"... Todorcevic showed... by employing a new technique — the *method of minimal walks* — which has also proved useful in many other applications.... This technique will be employed in this paper ...." A Solution to the L Space Problem (2004)

"The theorems in this paper are consequences of an analysis of coherent sequences of finite-to-one functions...and lower trace functions.... These are combinatorial objects which can be routinely constructed using the method of minimal walks." A Solution to the L Space Problem (2004)

A PRIMER ON THE METHOD OF MINIMAL WALKS

by

William Russell Hudson

A thesis submitted in partial fulfillment

of the requirements for the degree of

Master of Science in Mathematics

Boise State University

May, 2007

ABSTRACT

The method of minimal walks was developed over twenty years ago by Stevo Todorcevic and first expounded in his paper "Partitioning pairs of countable ordinals". An "updated, expanded, and clearer account" of the method is given in Todorcevic's "Coherent Sequences" chapter of the Handbook of Set Theory. Recently the method of minimal walks has played a seminal role in Justin Moore's construction of an L space and a background role in the consistency of a five element basis for the class of uncountable linear orders.

ABSTRACT

As a primer for the method of minimal walks this thesis intends to

(i) give the basic definitions needed to apply the method of minimal walks,

(ii) establish certain key facts that follow from the definitions,

(iii) demonstrate the application of the method by using it to construct certain objects with ‘nice’ properties.

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A *fundamental sequence,* or C*-sequence,* is a system of sets $(C_\alpha)_{\alpha<\omega_1}$ such that

(a) $C_{\alpha+1} = \{\alpha\}$,

(b) If α is a countable limit ordinal > 0, then C_α is an unbounded subset of α of order-type ω.

Note: (b) can also be phrased as

(b′) If α is a countable limit ordinal > 0, then C_α is a cofinal subset of α and if $\gamma < \alpha$ then $C_\alpha \cap \gamma$ is finite.

A *step* from a countable ordinal β towards a smaller ordinal α is the minimal point of C_β that is $\geq \alpha$. The cardinality of the set $C_\beta \cap \alpha$ is the *weight* of the step.

A *walk* (or *minimal walk*) from a countable ordinal β to a smaller ordinal α is the sequence $\beta = \beta_0 > \beta_1 > \ldots > \beta_n = \alpha$ such that for each $i < n$, the ordinal β_{i+1} is the step from β_i towards α.

The walk from β to α

$$\beta^0 = \beta$$

$$\beta^1 = \min(C_{\beta^0} \setminus \alpha)$$

$$\beta^2 = \min(C_{\beta^1} \setminus \alpha)$$

$$\beta^3 = \min(C_{\beta^2} \setminus \alpha)$$

$$\vdots$$

$$\beta^{n-1} = \min(C_{\beta^{n-2}} \setminus \alpha)$$

$$\beta^n = \alpha$$

$$C_{\beta^0} \cap \alpha$$

$$C_{\beta^1} \cap \alpha$$

$$\vdots$$

$$C_{\beta^n} \cap \alpha$$

The *upper trace* of the walk from β to α is the set

$$\mathsf{Tr}(\alpha,\alpha) = \{\alpha\},$$
$$\mathsf{Tr}(\alpha,\beta) = \{\beta\} \cup \mathsf{Tr}\Big(\alpha, \min\Big(C_\beta \setminus \alpha\Big)\Big).$$

The *lower trace* of the walk from β to α is the set

$$L(\alpha,\alpha) = \emptyset,$$
$$L(\alpha,\beta) = L\Big(\alpha, \min\Big(C_\beta \setminus \alpha\Big)\Big) \cup \Big\{\max\Big(C_\beta \cap \alpha\Big)\Big\} \setminus \max\Big(C_\beta \cap \alpha\Big).$$

The *full code* of the walk from β to α is the function $\rho_0 : [\omega_1]^2 \longrightarrow \omega^{<\omega}$ defined recursively by

$$\rho_0(\alpha, \alpha) = \langle \rangle ,$$
$$\rho_0(\alpha, \beta) = \left\langle \left| C_\beta \cap \alpha \right| \right\rangle ^\frown \rho_0\left(\alpha, \min\left(C_\beta \setminus \alpha\right)\right) .$$

The *maximal weight* of the walk is the two-place function $\rho_1 : [\omega_1]^2 \longrightarrow \omega$ defined recursively by

$$\rho_1(\alpha, \alpha) = 0,$$
$$\rho_1(\alpha, \beta) = \max\left\{ \left| C_\beta \cap \alpha \right| , \rho_1\left(\alpha, \min\left(C_\beta \setminus \alpha\right)\right) \right\} .$$

The function $\rho : [\omega_1]^2 \longrightarrow \omega$ is defined recursively by

$$\rho(\alpha, \alpha) = 0,$$
$$\rho(\alpha, \beta) = \max\{ \left| C_\beta \cap \alpha \right| , \rho\left(\alpha, \min\left(C_\beta \setminus \alpha\right)\right) , \rho(\xi, \alpha) : \xi \in C_\beta \cap \alpha \}.$$