A CANONICAL COUNTRYMAN LINE

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of the requirements for the degree of
Master of Science in Mathematics
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Outline

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- A Primer on Countryman Lines

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- A Primer on the Method of Minimal Walks

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- A Primer on Countryman Lines
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- Construction of a Countryman Line Using the Method of Minimal Walks

Countryman Lines

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- Real types, ω_1 types and ω_1^* types are not Countryman.

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- Todorcevic's full lower trace of the walk from β to α is defined recursively by

$$F(\alpha,\alpha)=\{\alpha\},$$
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- Our full lower trace of the walks from β and α is the set

$$F(\beta, \alpha) = \{\xi \leq \alpha : \mathrm{Tr}(\alpha, \xi) \cap \mathrm{Tr}(\beta, \xi) = \{\xi\}\}.$$

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- We use $<_{lex}$ for the right lexicographic ordering for finite sequences of integers, i.e.

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- We define the ordering $<_{\rho_0}$ on ω_1 by

$$\alpha <_{\rho_0} \beta \Leftrightarrow \rho_0(\alpha, \delta) <_{lex} \rho_0(\beta, \delta),$$

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$$\delta = \Delta(\alpha, \beta) = \min\{\zeta < \min\{\alpha, \beta\} : \rho_0(\alpha, \zeta) \neq \rho_0(\beta, \zeta)\}.$$

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- The uncountable linear order $C (\rho_0)$ is defined to be $(\omega_1, <_{\rho_0})$.

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Theorem: $C\left(\rho_0\right)$ is Countryman.

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- Define $\sigma : [\omega_1]^2 \to \left((\omega^{<\omega})^2\right)^{<\omega}$ by

$$\sigma(\alpha, \beta)(i) = (\rho_0(\alpha, \xi_i), \rho_0(\beta, \xi_i)),$$

where $(\xi_i)_{i=0}^{n}$ is the increasing enumeration of $F(\beta, \alpha)$.

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- For $\tau \in \left((\omega^{<\omega})^2\right)^{<\omega}$ define Γ_τ by

$$\Gamma_\tau = \{(\alpha, \beta) : \alpha < \beta < \omega_1 \; and \; \sigma(\alpha, \beta) = \tau\}.$$

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Lemma: For every $\tau \in \left((\omega^{<\omega})^2\right)^{<\omega}$, Γ_τ is a chain under the product order.

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For the complete proof and references please see

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That's it, thanks!

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