A Canonical Countryman Line

Boise Extravaganza in Set Theory

March, 2007

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Theorem (Moore) (PFA) The orders X, ω_1, ω_1^*, C and C^* form a five element basis for the uncountable linear orders any time X is a set of reals of cardinality $\aleph_1$ and C is a Countryman line.

Definition A *Countryman line* is an uncountable linear order whose Cartesian square is the union of countably many chains under the product order.

Fact If $C = (C, <_C)$ is Countryman, then $C^* = (C, >_C)$ is Countryman.

Fact If $C = (C, <_C)$ is Countryman and $L = (L, <_L)$ embeds into C and into C^*, then L is countable.

Fact Real types, ω_1 types and ω_1^* types are not Countryman.

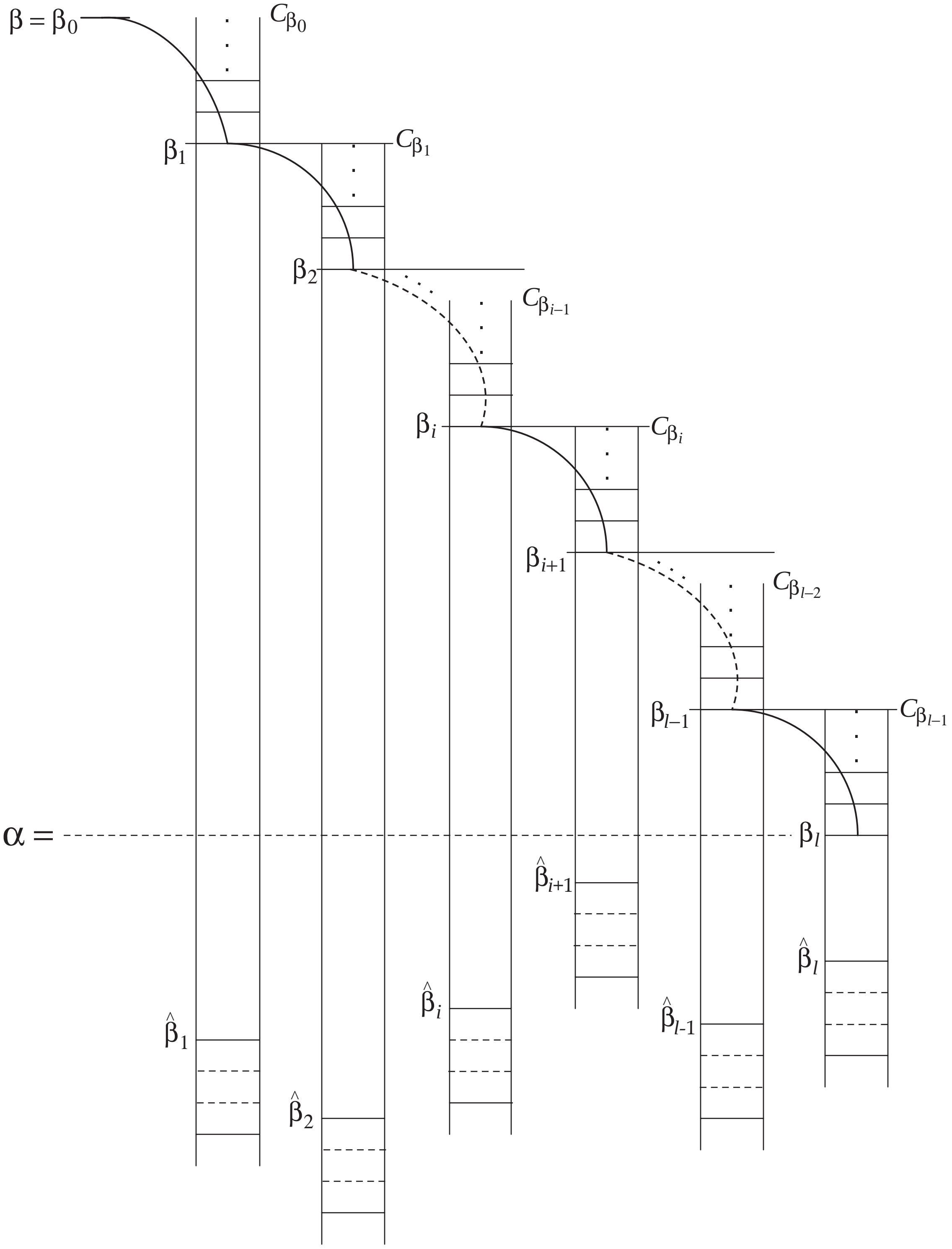

Figure 1. The Walk from β to α

Definition The *upper trace* of the walk from β to α is the function $\mathsf{Tr} : [\omega_1]^2 \to \mathcal{P}(\omega_1)$ given by

$$\mathsf{Tr}(\beta, \alpha) = \{\beta = \beta_0 > \beta_1 > \ldots > \beta_l = \alpha\},$$

where $\beta_0 = \beta$ and $\beta_i = \min\left(C_{\beta_{i-1}} \setminus \alpha\right)$.

Definition The *full code* of the walk is the function $\rho_0 : [\omega_1]^2 \to \omega^{<\omega}$ given by

$$\rho_0(\alpha, \alpha) = \emptyset,$$
$$\rho_0(\beta, \alpha)(i) = \left|C_{\beta_i} \cap \alpha\right|.$$

Definition The *full lower trace* of the walks from β and α is the set

$$F(\beta, \alpha) = \{\xi \leq \alpha : \mathsf{Tr}(\alpha, \xi) \cap \mathsf{Tr}(\beta, \xi) = \{\xi\}\}.$$

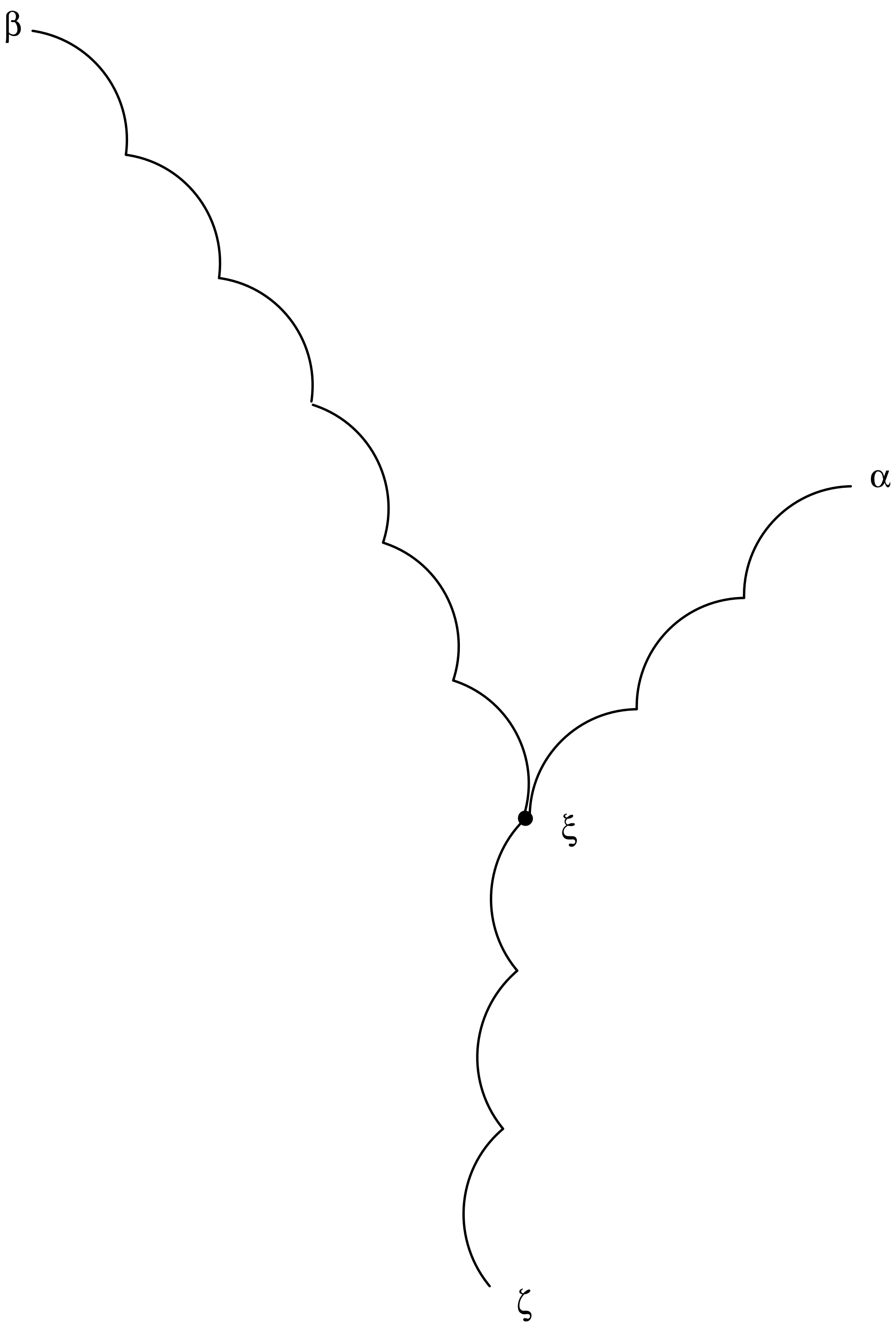

Figure 2. $\xi \in \wedge(\alpha,\beta)$

Definition We use $<_{lex}$ for the *right lexicographic ordering* for finite sequences of natural numbers, that is $s <_{lex} t$ if $t \sqsubset s$ or $s(j) < t(j)$ where j is the least i such that $s(i) \neq t(i)$.

Definition For $\alpha, \beta \in \omega_1$ we define

$$\alpha <_{\rho_0} \beta \ if \ and \ only \ if \ \rho_0(\alpha, \delta) <_{lex} \rho_0(\beta, \delta),$$

where

$$\delta = \Delta(\alpha, \beta) = \min\{\zeta < \min\{\alpha, \beta\} : \rho_0(\alpha, \zeta) \neq \rho_0(\beta, \zeta)\}.$$

Definition The uncountable linear order $C(\rho_0)$ is defined to be $\left(\omega_1, <_{\rho_0}\right)$.

Theorem $C(\rho_0)$ is a Countryman type.

Definition Define $\sigma : [\omega_1]^2 \to \left(\left(\omega^{<\omega} \right)^2 \right)^{<\omega}$ by

$$\sigma(\alpha, \beta)(i) = (\rho_0(\alpha, \xi_i), \rho_0(\beta, \xi_i)),$$

where $(\xi_i)_{i=0}^{n}$ is the increasing enumeration of $F(\beta, \alpha)$.

Definition For $\tau \in \left(\left(\omega^{<\omega} \right)^2 \right)^{<\omega}$ define Γ_τ by

$$\Gamma_\tau = \{(\alpha, \beta) : \alpha < \beta < \omega_1 \ and \ \sigma(\alpha, \beta) = \tau\}.$$

Theorem Suppose

$$(\alpha, \beta), \left(\alpha', \beta' \right) \in \Gamma_\tau.$$

Then

$$\alpha <_{\rho_0} \alpha' \ if \ and \ only \ if \ \beta <_{\rho_0} \beta'.$$

Proof Let $(\xi_i)_{i=0}^{n}$ and $(\xi'_i)_{i=0}^{n}$ be the respective increasing enumerations of $F(\alpha, \beta)$ and $F(\alpha', \beta')$ and let j be the least i such that $\xi_i \neq \xi'_i$, assuming without loss of generality that $\xi_j < \xi'_j$. By assumption for $i \leq n$

$$\rho_0(\alpha, \xi_i) = \rho_0\left(\alpha', \xi'_i\right) \; and \; \rho_0(\beta, \xi_i) = \rho_0\left(\beta', \xi'_i\right).$$

Key Fact If $\zeta < \alpha < \beta < \omega_1$ and $\xi = \min(F(\beta, \alpha) \setminus \zeta)$, then

$$\begin{aligned} \rho_0(\alpha, \zeta) &= \rho_0(\alpha, \xi) \frown \rho_0(\xi, \zeta), \\ \rho_0(\beta, \zeta) &= \rho_0(\beta, \xi) \frown \rho_0(\xi, \zeta). \end{aligned}$$

Set $\delta_1 = \mathbf{\Delta}(\alpha, \alpha')$ and $\delta_2 = \mathbf{\Delta}(\beta, \beta')$. First,

$$\begin{aligned}
\rho_0\left(\alpha', \xi_j\right) &= \rho_0\left(\alpha', \xi'_j\right) \frown \rho_0\left(\xi'_j, \xi_j\right) \\
&= \rho_0\left(\alpha, \xi_j\right) \frown \rho_0\left(\xi'_j, \xi_j\right) \neq \rho_0\left(\alpha, \xi_j\right) \\
&\Rightarrow \delta_1 \leq \xi_j.
\end{aligned}$$

Likewise, $\delta_2 \leq \xi_j$.

Second, if $\zeta \leq \xi_{j-1} = \xi'_{j-1}$, then there is $k < j$ such that

$$\begin{aligned}
\rho_0(\alpha, \zeta) &= \rho_0(\alpha, \xi_k) \frown \rho_0(\xi_k, \zeta) \\
&= \rho_0\left(\alpha', \xi'_k\right) \frown \rho_0\left(\xi'_k, \zeta\right) = \rho_0\left(\alpha', \zeta\right) \\
&\Rightarrow \delta_1 > \xi_{j-1}.
\end{aligned}$$

Likewise, $\delta_2 > \xi_{j-1}$.

Since $\delta_1, \delta_2 \in \left(\xi_{j-1}, \xi_j\right]$.

$$\begin{aligned}\rho_0(\alpha, \delta_1) &= \rho_0\left(\alpha, \xi_j\right) \frown \rho_0\left(\xi_j, \delta_1\right) \\ &\neq \rho_0\left(\alpha', \xi'_j\right) \frown \rho_0\left(\xi'_j, \delta_1\right) = \rho_0\left(\alpha', \delta_1\right),\end{aligned}$$

hence

$$\begin{aligned}\rho_0(\beta, \delta_1) &= \rho_0\left(\beta, \xi_j\right) \frown \rho_0\left(\xi_j, \delta_1\right) \\ &\neq \rho_0\left(\beta', \xi'_j\right) \frown \rho_0\left(\xi'_j, \delta_1\right) = \rho_0\left(\beta', \delta_1\right),\end{aligned}$$

so

$$\delta_2 \leq \delta_1.$$

Similarly, $\delta_1 \leq \delta_2$ so that

$$\delta_1 = \delta_2.$$

Let $\delta = \delta_1 = \delta_2$. Then

$$\begin{aligned} \rho_0(\alpha, \delta) &= \rho_0\left(\alpha, \xi_j\right) \frown \rho_0\left(\xi_j, \delta\right), \\ \rho_0\left(\alpha', \delta\right) &= \rho_0\left(\alpha', \xi_j'\right) \frown \rho_0\left(\xi_j', \delta\right), \end{aligned}$$

and

$$\begin{aligned} \rho_0(\beta, \delta) &= \rho_0\left(\beta, \xi_j\right) \frown \rho_0\left(\xi_j, \delta\right), \\ \rho_0\left(\beta', \delta\right) &= \rho_0\left(\beta', \xi_j'\right) \frown \rho_0\left(\xi_j', \delta\right). \end{aligned}$$

Thus

$$\begin{multlined} \rho_0(\alpha, \delta) <_{lex} \rho_0\left(\alpha', \delta\right) \Leftrightarrow \\ \rho_0\left(\xi_j, \delta\right) <_{lex} \rho_0\left(\xi_j', \delta\right) \Leftrightarrow \\ \rho_0(\beta, \delta) <_{lex} \rho_0\left(\beta', \delta\right). \end{multlined}$$