Symmetry motivates a new consistent fragment of NF and an extension of NF with semantic motivation

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Abstract

A sequence of theories of sets and classes is presented, indexed by a natural number parameter k. The criterion for determining which classes are sets has to do with symmetry. The theories with $k \geq 2$ are shown to entail Quine's New Foundations, which is interesting as the criteria for sethood do not involved typed set theory or syntax of formulas. The theories with $k = 0, 1$ are shown to be consistent. The case $k = 0$ is related to recent work of Sergei Tupailo. The case $k = 1$ yields a consistency proof for a new consistent subtheory of New Foundations extending the theory recently described by Tupailo and the theory NF_3 shown to be consistent by Grishin in 1969.

Contents

1 Introduction

In this paper we introduce a family of theories of sets and classes motivated by symmetry considerations, indexed by a natural number parameter. For the smallest nontrivial value of the parameter we give a consistency proof of the associated theory, and we show that the sets of this theory satisfy a fragment of *NF* not heretofore known to be consistent, extending both the theory NF_3 shown to be consistent by Grishin ([2]) and the theory *NFSI* defined and shown to be consistent by Sergei Tupailo ([10]). For each larger value of the parameter we obtain a theory whose sets satisfy *NF*, and whose comprehension axioms have a semantic rather than syntactical character. We have published a paper on a similar but much more complicated extension of *NF* earlier ([4]).

2 Basics of the Set/Class Theory

The theories we present are first-order theories with equality and membership. General objects are called classes.

extensionality: $(\forall xy.x = y \leftrightarrow (\forall z.z \in x \leftrightarrow z \in y))$ is an axiom.

definition of sethood: $\texttt{set}(x) \equiv_{\texttt{def}} (\exists y.x \in y)$.

predicative class comprehension: For each formula ϕ in which each quantifier is restricted to a class, and in which the variable A does not occur free, $(\exists A.(\forall x.x \in A \leftrightarrow \texttt{set}(x) \wedge \phi))$. The object A witnessing this (unique by extensionality) is called $\{x \in V \mid \phi\}$. V is defined as $\{x \in V \mid x = x\}$.

Thus far we have an entirely standard predicative theory of classes. The interesting question in such a theory is the criterion for sethood. We give axioms which provide some specific sets; the main set comprehension axiom appears below.

empty set: The class $\{x \in V \mid x \neq x\}$ is a set. We call it $\emptyset$.

pairing: For any sets a and b, the class $\{a, b\} \equiv_{\texttt{def}} \{x \in V \mid x = a \vee x = b\}$ is a set. $\{a\}$ denotes $\{a, a\}$.

definition of ordered pair: The usual notation $(a, b) \equiv_{\texttt{def}} \{\{a\}, \{a, b\}\}$ is introduced. This is proved to be an ordered pair in the standard way. Relations and functions are defined in the standard way, and the ability to implement relations on sets defined by bounded formulas as classes follows from pairing and predicative comprehension.

3 Image Operations, Permutations and Symmetry

If f is a class function and A is a class, we define $f``A$ as $\{y \in V \mid (\exists x \in A.y = f(x))\}$ as usual. Further, we define the function $f_1$: $y = f_1(x)$ iff $x$ and $y$ are sets and $y = f``x$. We note that f_1 may fail to be defined (as a function) for some sets, but we reserve the right to use the notation $f_1(x)$ for the class $f``x$ even when it is not a set.

We define $f_0(x)$ as $f(x)$, for any class function f. For each concrete natural number n, we define $f_{n+1}(x)$ as $f_n``(x) = (f_n)_1(x)$, with the same remark that f_{n+1} might not be defined as a function at some sets x, but we nonetheless use the notation $f_{n+1}(x)$ for the appropriate class. We do not use the notation f_{n+1} unless f_n is a total function.

A bijection from V to V is called a *permutation*. Note that if f is a permutation and f_1 is total, it follows that f_1 is also a permutation. We call a permutation f n-setlike iff f_n is a permutation (equivalently, if it is total). We call a permutation *setlike* iff it is n-setlike for each n, but we have logical difficulties with actually saying this.

We can now state a defective version of our set comprehension axiom (with a concrete natural number parameter k).

k-symmetric comprehension: A class A is a set iff there is a finite set S such that for any k-setlike permutation such that $f_k(s) = s$ for each $s \in S$, we have $f_{k+1}(A) = A$. The finite set S is called a *support* of A.

The defect is the presence of the word "finite", which we have not defined. For the smallest values of k, the best we can do is introduce "meta-finite" as a primitive notion and add some axioms regulating it, and the further stipulation that we do not add instances of class comprehension mentioning the meta-finiteness predicate. We call this meta-finiteness because there is no reason to believe it will coincide with the notion of finiteness as interpreted in the set theory we are developing.

empty set: The empty set is meta-finite.

pairs: For any sets x, y, $\{x, y\}$ is a meta-finite set.

union: The union of a meta-finite set of meta-finite sets is meta-finite.

collection: Let ϕ be any formula (it does not need to be bounded and it may mention the meta-finiteness predicate). If A is meta-finite and $(\forall x \in A.(\exists y.\phi(x, y)))$ then there is a meta-finite set B such that $(\forall x \in A.(\exists y \in B.\phi(x, y)))$ and moreover $(\forall y \in B.(\exists x \in A.\phi(x, y)))$.

k-symmetric comprehension: A class A is a set iff there is a meta-finite set S such that for any k-setlike permutation such that $f_k(s) = s$ for each $s \in S$, we have $f_{k+1}(A) = A$. The meta-finite set S is called a *support* of A.

For k at least 4 we can demonstrably take the approach that supports in the sense above are assumed to have one element, and so have no need to introduce the primitive meta-finiteness notion. This approach may work for $k = 2$ and $k = 3$ as well, as we will see.

k-symmetric comprehension (one-point support version, for $k \geq 2$): A class A is a set iff there is a set S such that for any k-setlike permutation such that $f_k(S) = S$, we have $f_{k+1}(A) = A$. The set $\{S\}$ is called a *support* of A; S itself may be called the "support element". If we were not working in parallel with the definition above, we would call S itself the support here.

4 Stratification and Permutations

Definition: A formula ϕ in the language of equality and membership is said to be *stratified* iff there is a function σ (called a *stratification* of ϕ) from the variables appearing in ϕ (considered as syntactical objects) to natural numbers such that for each subformula $x = y$ of ϕ we have $\sigma(``x") = \sigma(``y")$. and for each subformula $x \in y$ of ϕ we have $\sigma(``x") + 1 = \sigma(``y")$. Hereinafter we will write $\sigma(x)$ instead of the very careful $\sigma(``x")$.

A formula is stratified iff a suitable assignment of types to its variables (coded by the stratification) would make it a well-formed formula of the simple type theory of sets. But as we will see this concept can be used effectively in an untyped theory of sets and classes. The notion of stratification is of course from [7].

We prove a Lemma about the action of permutations on classes with stratified definitions. This result is adapted from results of Coret relating stratified formulas and permutations, found in [1].

Lemma: Let $\phi(x, a_1, \ldots, a_n)$ be a stratified formula (which we will sometimes call just ϕ) in which any free variable that appears is x or one of the a_i's, let σ be a stratification of ϕ, and let f be a permutation which is $\sigma(y)$-setlike for each variable y appearing free or bound in ϕ: it follows that

$$f_{\sigma(x)+1}(\{x \mid \phi(x, a_1, \ldots, a_n)\}) = \{x \mid \phi(x, f_{\sigma(a_1)}(a_1), \ldots, f_{\sigma(a_n)}(a_n))\}.$$

Proof: Observe that for any u and v, $u = v \leftrightarrow f_k(u) = f_k(v)$ if f is k-setlike (or even $(k-1)$-setlike), and $u \in v \leftrightarrow f_0(u) \in f_1(v) \leftrightarrow f_1(u) \in f_2(v) \leftrightarrow \ldots \leftrightarrow f_k(u) \in f_{k+1}(v)$, if f is k-setlike.

Let ϕ_1 be obtained from ϕ by replacing each atomic subformula $u\,R\,v$ of ϕ with $f_{\sigma(u)}(u)\,R\,f_{\sigma(v)}(v)$, where R is the equality or membership predicate. Because σ is a stratification, we have $\sigma(u) = \sigma(v)$ if R is equality and $\sigma(u) + 1 = \sigma(v)$ if R is membership. In either case by considerations in the previous paragraph we have $u\,R\,v$ equivalent to $f_{\sigma(u)}(u)\,R\,f_{\sigma(v)}(v)$, so we have ϕ_1 equivalent to ϕ. Notice that each variable u appears in ϕ_1 only in the context $f_{\sigma(u)}(u)$.

Now we observe that each subformula $(\forall u.\psi(f_{\sigma(u)}(u)))$ of ϕ_1 is precisely equivalent to $(\forall u.\psi(u))$, because $f_{\sigma(u)}$ is a permutation of the universe. We can carry out this transformation on each quantified subformula to obtain the formula $\phi(f_{\sigma(x)}(x), f_{\sigma(a_1)}(a_1), \ldots, f_{\sigma(a_n)}(a_n))$, which is thus seen to be equivalent to the original formula ϕ. It then follows that

$$f_{\sigma(x)+1}(\{x \mid \phi(x, a_1, \ldots, a_n)\}) = \{x \mid \phi(x, f_{\sigma(a_1)}(a_1), \ldots, f_{\sigma(a_n)}(a_n))\} :$$

$u \in f_{\sigma(x)+1}(\{x \mid \phi(x, a_1, \ldots, a_n)\})$ is equivalent to

$$(\exists v.u = f_{\sigma(x)}(v) \wedge \phi(v, a_1, \ldots, a_n)),$$

which is equivalent to

$$(\exists v.u = f_{\sigma(x)}(v) \wedge \phi(f_{\sigma(v)}(v), f_{\sigma(a_1)}(a_1), \ldots, f_{\sigma(a_n)}(a_n))),$$

by the argument above, which is simply equivalent to

$$\phi(u, f_{\sigma(a_1)}(a_1), \ldots, f_{\sigma(a_n)}(a_n)).$$

5 Setlike is k-Setlike

In this section we prove that in the theory with k-symmetric comprehension, the notion "k-setlike" is effectively equivalent to "setlike". We do not really prove this because we cannot really even say it. But the theorem we now state and prove has this effect on an informal level.

Theorem: It follows from k-symmetric comprehension that if f is a k-setlike permutation, f_1 is also a k-setlike permutation. (Note that this implies that f_n is a k-setlike permutation for each concretely given natural number n.)

Proof: Let f be a k-setlike permutation. This means that for any set A, $f_k(A)$ is a set. Our aim is to show that for any set A, $(f_1)_k(A) = f_{k+1}(A)$ is a set. So let A be a set with support S. We recall that this means that for any k-setlike permutation g such that $g_k(s) = s$ for each $s \in S$, we have $g_{k+1}(A) = A$.

We claim that $f_{k+1}(A)$ is a set with support $f_{k+1}(S)$. $f_{k+1}(S)$ is meta-finite by an application of collection for meta-finite sets. Let g be a

k-setlike map such that $g_k(s) = s$ for each $s \in f_{k+1}(S)$. It follows that $(gf)_k(s) = f_k(s)$ for each $s \in S$, so $(f^{-1}gf)_k(s) = s$ for each $s \in S$, from which it follows that $(f^{-1}gf)_{k+1}(A) = A$, from which it follows that $(gf)_{k+1}(A) = g_{k+1}(f_{k+1}(A)) = f_{k+1}(A)$, which completes the proof that $f_{k+1}(A)$ is a set with support $f_{k+1}(S)$.

The proof for the one-point version is in effect this proof with the simplifying assumption that the support S has a single element s: the one mention of meta-finiteness above becomes unnecessary.

6 Towards Stratified Comprehension

In this section we prove that the extensions of certain stratified formulas are sets.

Theorem: If k-symmetric comprehension is assumed and $\phi(x, a_1, \ldots, a_n)$ is a stratified formula (sometimes briefly called ϕ) with stratification σ such that any free variable that appears in ϕ is either x or one of the a_i's and $\sigma(x) \leq k$, then $\{x \mid \phi\}$ is a set. The parameters a_i are taken here to be *sets*.

Proof: Note that we can assume that $\sigma(x) = k$, because if $\sigma(x) < k$ we can use the stratification $\sigma'(y) = \sigma(y) + k - \sigma(x)$ instead. We will make this assumption henceforth.

Let f be a k-setlike permutation (and therefore an m-setlike permutation for every concrete m). We then have $f_{k+1}(\{x \mid \phi(x, a_1, \ldots, a_n)\}) = \{x \mid \phi(x, f_{\sigma(a_1)}(a_1), \ldots, f_{\sigma(a_n)}(a_n))\}$. For f_{k+1} to fix $\{x \mid \phi\}$, it is sufficient for $f_{\sigma(a_i)}$ to fix a_i for each parameter a_i. For each i, we need a meta-finite set S_i such that if f is k-setlike and f_k fixes each element of S_i, it follows that $f_{\sigma(a_i)}$ fixes a_i: the union of the S_i's will then be a meta-finite set (because it is the union of a concretely given meta-finite set of meta-finite sets) such that if f_k fixes each element of S, then $f_{k+1}(\{x \mid \phi\}) = \{x \mid \phi\}$, that is a support witnessing the sethood of $\{x \mid \phi\}$.

If $\sigma(a_i) \leq k$, it is easy to find the appropriate S_i. Let ι represent the singleton operation on sets (so we can iterate it: note that we do take $\iota^0(u) = u$). $f_{\sigma(a_i)}$ fixes a_i exactly if f_k fixes $\iota^{k-\sigma(a_i)}(a_i)$, so we can define S_i as $\iota^{k-\sigma(a_i)+1}(a_i)$.

To complete the proof, we prove a Lemma.

Lemma (assuming k-symmetric comprehension): For any concretely given natural number n and any set A, there is a meta-finite set S_n such that for any k-setlike permutation f such that f_k fixes each element of S_n, $f_{k+n+1}(A) = A$.

Proof: This is proved by induction on n. One might think of this better as a concretely given sequence of proofs, of which we carry out only as many as we need. The basis step is trivial: it follows directly from k-symmetric comprehension.

Suppose that the Lemma has been demonstrated for $n = m$. We show that it follows for $n = m + 1$. Fix a set A. Let S_m witness the truth of the Lemma for A with $n = m$. For each element of the meta-finite set S_m there is a meta-finite set which is a support for it, so by collection for meta-finite sets there is a meta-finite set Σ of supports for elements of S_m which contains a support for each of the elements of S_m. We define S_{m+1} as the union of Σ (and we see that S_{m+1} is meta-finite, since it is the union of a meta-finite set of meta-finite sets). Now suppose that f is a k-setlike permutation such that f_k fixes each element of S_{m+1}. It follows that f_k fixes each element of the support of each element of S_m, so $f_{k+1} = (f_1)_k$ fixes each element of S_m, so $(f_1)_{k+m+1} = f_{k+(m+1)+1}$ fixes A, which is what we needed.

The lemma is exactly what is needed to define all the sets S_i needed to ensure that each parameter is fixed, so we are done.

Theorem: If the one-point support version of k-symmetric comprehension is assumed and $\phi(x, y)$ is a stratified formula (sometimes briefly called ϕ) with stratification σ such that any free variable that appears in ϕ is either x or y and $\sigma(x) \leq k$, then $\{x \mid \phi\}$ is a set.

Proof: The proof goes exactly as above, with the modification that the word "meta-finite" is replaced everywhere by "one-element" (which forces the restriction to a single parameter).

7 Arriving at Stratified Comprehension

We now consider the case $k \geq 2$, and for some purposes the case $k \geq 4$.

The theory *NF* (New Foundations) is the first-order theory with equality and membership as primitive predicates whose axioms are extensionality (objects with the same elements are the same) and stratified comprehension (for any stratified formula ϕ, $\{x \mid \phi\}$ exists). This theory was proposed by W. v. O. Quine in 1937 ([7]). It was shown to disprove AC by Specker in 1953 ([9]). *NF* is not known to be consistent; the scheme of stratified comprehension was shown by Jensen in [5] to be consistent with weak extensionality (objects with elements are equal iff they have the same elements) and moreover with Infinity and Choice.

The sets of the theory given here satisfy extensionality because the underlying class theory has the same extensionality axiom. The previous section proves that the sets of this theory satisfy a subscheme of the stratified comprehension scheme. Previously established results about *NF* show that we have proved more than we might suppose.

Theorem: The axiom scheme "for each stratified formula ϕ with stratification σ such that $\sigma(x) \leq 2$, $\{x \mid \phi\}$ exists" is equivalent to full stratified comprehension.

Proof: Grishin showed in [3] that the axiom scheme containing those assertions '$\{x \mid \phi\}$ exists' with stratification σ with codomain $\{0, 1, 2, 3\}$ is equivalent to full stratified comprehension. The scheme under consideration properly includes this one.

Corollary: The theory given here with the full k-symmetric comprehension axiom with $k \geq 2$ entails *NF* on the domain of sets.

Theorem: The axiom scheme of stratified comprehension is equivalent to the axiom of pairing plus the axiom of stratified comprehension with a single parameter.

Proof: Two parameters a and b in a stratified formula with stratification σ can be merged into one by replacing references to a with references to the suitably iterated element of the first projection of $(\iota^{\sigma(b)}(a), \iota^{\sigma(a)}(b))$, and references to b with references to the suitably iterated element of the second projection of the same object. Iteration of this (quite nasty) procedure will reduce all parameters to a single parameter. The only instance of comprehension with more than one parameter which is needed is the one which provides the ordered pair, and the axiom of pairing will do this (by allowing us to define the Kuratowski ordered pair).

Corollary: The theory developed here with the one-point support version of k-symmetric comprehension with $k \geq 2$ entails *NF* on the domain of sets.

There is a reason to prefer $k \geq 4$ if ones aim is to obtain a semantically motivated extension of *NF*. The difficulty has to do with the interaction between the axiom of unordered pairs and the one-point k-symmetric comprehension axiom. If we use the full k-symmetric comprehension axiom, the sethood of an arbitary unordered pair $\{a, b\}$ is quite believable: the support is simply $\{a, b\}$ itself: if f_k fixes a and f_k fixes b, then of course f_{k+1} fixes $\{a, b\}$. If we use the one-point version something more impressive is being said: for any a and b, there is a c such that if $f_k(c) = c$, it follows that f_{k+1} fixes $\{a, b\}$. For $k = 4$, such a c can be found in a uniform way. If we write the Quine ordered pair of a and b (defined in [8]) as $\langle a, b\rangle$, we can observe that it is straightforward to prove $f_4(\langle a, b\rangle) = \langle f_4(a), f_4(b)\rangle$, so f_4 fixes the Quine pair of a and b iff it fixes both a and b, from which it follows that the one-point version of the k-symmetric comprehension axiom says no more for $k \geq 4$ than the full version of the axiom. It is unclear to us as we write whether it is reasonable to suppose that unordered pairs of arbitrary sets have one-point supports for $k = 2$ or $k = 3$.

We have now arrived at one of the main conclusions of this paper: the first-order theory of sets and classes whose axioms are extensionality for classes, predicative class comprehension, empty set, pair set, and the assertion that "a class A is a set iff there is a set S such that for any 4-setlike class permutation f such that $f_4(S) = S$ we have $f_5(A) = A$" entails *NF* on the domain of sets.

This addresses one of the main criticisms of *NF*, that is the assertion that the stratified comprehension scheme is merely a syntactical trick. The theory presented here does not talk about syntax at all: it tells us what sets and classes are supposed to be like, and its criterion for sets is that they should be highly *symmetrical* in a particular way. Of course stratified comprehension already has known semantics, since we have models of *NFU*, the theory with weak extensionality shown to be consistent by Jensen. But here we have obtained a description of what the world of sets and classes might be like which would entail *NF* itself if it could be realized, with no allusion to syntax of formulas or to typed theory of sets in its formulation. We did something like this in [4], but the theory presented there is far more complicated.

8 Additional technical observations on the case $k \geq 2$

There is a version of set comprehension which implies empty set and pairing and also entails New Foundations:

Axiom of set comprehension (alternative form): A class C is a set iff there is a class S with either 0, 1, or 2 elements (a condition expressible in first-order logic) such that for any 4-setlike permutation f such that $f_4(s) = s$ for each $s \in S$, we also have $f_5(C) = C$.

Discussion: This implies empty set because $S = \emptyset$ will witness the condition if $C = \emptyset$. This implies pairing because $S = \{x, y\}$ will witness the condition if $C = \{x, y\}$. Finally, if a class C has a support $S = \{x, y\}$ with two elements, it also has the support $\{(x, y)\}$ (where (x, y) is the Quine pair of x and y) with one element. The Quine pair (x, y) is easily seen to be a set with support $\{x, y\}$. So, we can formulate a theory entailing *NF* with the axioms Extensionality, Predicative Class Comprehension, and Set Comprehension as formulated here.

We have two observations to make about the theory of the ordinals.

Ordinal numbers are defined in *NF* as equivalence classes of well-orderings under similarity. Of course, the notion of well-ordering proper to *NF* is "a linear order in which each sub*set* of the domain has a least element". We might want to consider the *true* well-orderings, those linear orders in which each sub*class* of the domain has a least element. If we code a well-ordering as the set of its initial segments, we find that the class of all true well-orderings, if it were definable, would be 3-symmetric, and therefore a set. Working in the interpreted *NF*, we could then establish that the set of true ordinals existed (we could also see this directly as it would be a 4-symmetric class), and the Burali-Forti paradox would follow at once from consideration of the first non-true ordinal (the order type of the natural well-ordering of the ordinals restricted to the true ordinals). From this contradiction, it follows that the class of true well-orderings does not exist. This does not contradict our axiom of class comprehension, because the formula defining the class of true ordinals is impredicative. This result extends known "pathologies" of the ordinals in *NF*; it was already known that they are not externally well-ordered. This is the reason why we use predicative class comprehension in the formulation of our theory. The use of the particular coding of well-orderings as sets of initial segments is not significant here; it allows us to finesse considerations of what definition of the ordered pair we are using when determining the degree of symmetry of the classes under discussion.

Our other observation about the ordinals we make very briefly and without full details, as this would take us into technical complexities little-known except to specialists in "New Foundations" and related theories. It is really in the nature of a hint to such specialists to go work out the details, which are not difficult in the presence of specialized knowledge which it would take too long to present here. An ordinal α is called *strongly cantorian* iff for any well-ordering belonging to α, the restriction of the singleton map to the domain of α is a set function. Sufficiently large ordinals (such as the order type of the natural well-ordering of the set of ordinals) are not strongly cantorian. The class of strongly cantorian ordinals is not a set; it definitely exists as a class as it has a predicative definition. It is straightforward to show that if as above we code a well-ordering as the set of its initial segments, the class of strongly cantorian ordinals is 4-symmetric with respect to set permutations. But as it cannot be a set, this implies that in the theory discussed here with $k \geq 2$, there must be a setlike permutation f such that $j^4(f)$ moves a strongly cantorian ordinal α to a non-strongly-cantorian ordinal. A little further work shows that for such an ordinal α, $j^i(f)(\alpha)$ is distinct from α for every i: such an ordinal is not i-symmetric for any i. This contrasts with the situation in the theory with $k = 1$ for which we produce a model in the last section of this paper: in that model, every set is i-symmetric for some i. Further analysis of the consequences of the non-sethood of the class of strongly cantorian ordinals reveals considerable complexities in the setlike permutations of the ordinals, which may hint at what a model of this theory is like or at an eventual proof of inconsistency.

9 The case $k = 0$ and Sergei Tupailo's *NFSI*

In this section we consider the version of our theory of sets and classes with $k = 0$. The axiom of set comprehension is

0-symmetric comprehension: A class A is a set iff there is a meta-finite set S such that for any 0-setlike permutation f such that $f(s) = s$ for each $s \in S$, we have $f_1(A) = A$. The meta-finite set S is called a *support* of A.

First observe that any permutation at all is 0-setlike: this means that for any set x, $f_0(x) = f(x)$ is a set, and this is true for any permutation f.

Suppose that A is a set with meta-finite support S. Then A must be either the universe, the empty set, S, or S^c. Note that for any two sets x and y, the permutation (xy) which transposes x and y and fixes everything else exists as a class. This implies that A and A^c cannot both have nonempty intersection with S: if $x \in A \cap S$ and $y \in A^c \cap S$, then (xy) fixes S elementwise but does not fix A. For the same reason A and A^c cannot both have nonempty intersection with S^c. But this implies immediately that A is either the universe, the empty set, S or S^c.

If S is meta-finite, clearly S and S^c are sets by the criterion above. So the set comprehension axiom with $k = 0$ is precisely equivalent to the assertion that a class is a set iff it is either meta-finite or the complement of a meta-finite set.

It is easy to establish that this theory has a model. Let (M, E) be a structure for the language of set theory with the property that the preimages under E of the elements of M are exactly the finite and cofinite subsets of M, and that distinct elements of M have distinct preimages. This can be converted to a structure satisfying class comprehension by adding all the infinite and co-infinite subsets of M as new elements, extending E so that the E-preimage of each new element is its extension as a set. This structure will satisfy extensionality and predicative (indeed impredicative) class comprehension. It clearly satisfies pairing. If "meta-finite" is interpreted as "finite", all the axioms of meta-finiteness are satisfied. We then observe that sethood is equivalent to being either meta-finite or co-meta-finite, and we have seen that this is equivalent to our set comprehension axiom with $k = 0$.

There are such structures (M, E). For example, let M be the set of natural numbers and define $m \, E \, n$ as holding if either $n = 2k$ and the mth binary digit of k is 1 or $n = 2k + 1$ and the mth binary digit of k is 0. It will be evident that every finite set of natural numbers is the E-preimage of a unique even natural number and every cofinite set of natural numbers is the E-preimage of a unique odd natural number.

Further, the results above show that this structure satisfies extensionality and the restricted version of stratified comprehension in which $\{x \mid \phi\}$ is asserted to exist if ϕ is stratified with a stratification σ such that $\sigma(x) = 0$ (that is, the relative type assigned to x is the lowest type assigned to any variable in ϕ). This theory *NFSI* (for "strongly impredicative *NF*") was defined and shown to be consistent by Sergei Tupailo using a far more complex argument involving forcing ([10]); the simpler class of models of this theory given here was noted subsequently by Marcel Crabbé (personal communication). Tupailo's models of *NFSI* satisfy some additional axioms not satisfied by the simpler models, such as the existence of each Frege natural number.

Although the consistency proof of *NFSI* originally given by Tupailo is enormously more complicated than the proof due to Crabbé which is incorporated here into our framework of symmetric set/class theories, we wish to carefully avoid any implication of a slight to Tupailo's contribution. Tupailo's models satisfy some additional assertions of interest, and it is also notable that no one noticed the subsystem *NFSI* earlier, though it does have a very direct consistency proof. The formulation of *NFSI* then suggests the theory *NFMSI* considered in the next section: without Tupailo's work the present paper would not have been written.

10 The case $k = 1$: the new theory *NFMSI*

NFMSI (moderately strongly impredicative *NF*), a new theory proposed here, is the fragment of *NF* consisting of strong extensionality and those instances of comprehension "$\{x \mid \phi\}$ exists" in which ϕ can be stratified in a way which assigns x the lowest or second lowest type. This theory obviously extends *NFSI* and also extends a long-known consistent subsystem of *NF*, the system NF_3 whose axioms are extensionality and those axioms "$\{x \mid \phi\}$ exists" which have a stratification with range $\{0, 1, 2\}$, defined and shown to be consistent by Grishin in [2]. It is important to note that the actual form of "$\{x \mid \phi\}$ exists" is $(\exists A.(\forall x.x \in A \leftrightarrow \phi))$, from which we can see that the type assigned to x by the stratification must be one lower than the type assigned to A, so either 0 or 1, so each such axiom is also an axiom of *NFMSI*.

Results above show that *NFMSI* is entailed by the theory with sets and classes whose axioms are extensionality for classes, existence of the empty set and ordered pairs, predicative class comprehension, and the set comprehension axiom of the first part with $k = 1$, namely

Symmetric set comprehension ($k = 1$): A class A is a set iff there is a metafinite set S such that for every 1-setlike permutation f such that $f_1(s) = s$ for each $s \in S$, we have $f_2(A) = A$.

We will construct a model of this theory. We stipulate that the meaning of "metafinite" will be exactly "finite" (which will certainly meet the requirements placed on the notion of metafiniteness).

Definition: A *relative cardinal* is a pair of cardinals (κ, λ). The cardinality of a set A relative to a set B, written $|A|_B$, is $(|A \cap B|, |B - A|)$. $(\kappa, \lambda) + (\kappa', \lambda') = (\kappa + \kappa', \lambda + \lambda')$ defines addition of relative cardinals.

Definition: Let X be a set and let Y be a subset of $\mathcal{P}(X)$ which is closed under finite intersections and relative complement with respect to X.

Let S be a finite subset of Y. We define $V(S)$ [in a context where the identities of X and Y are understood] as the collection of minimal nonempty elements in the inclusion order of the smallest superset of S which is closed under finite intersections and complement with respect to X (the Venn diagram of S). Note that $V(\emptyset) = \{X\}$.

We say that A is (X, Y)-symmetric with support S iff $A \subseteq Y$ and for each $a, b \in Y$, $a \in A \leftrightarrow b \in A$ if for each $T \in V(S)$ we have $|a|_T = |b|_T$. A is (X, Y)-symmetric with support S if membership of any $y \in Y$ in A is decided by the relative cardinalities of y relative to the elements of the Venn diagram of S.

In this situation, we define g_a^S for each $a \in Y$ as the function sending each $T \in V(S)$ to $|a|_T$. Notice that from the domain of such a function we can determine what X is and what $V(S)$ is. Notice that a subset A of Y is (X, Y)-symmetric with support S iff for all $a, b \in Y$ we have $g_a^S = g_b^S \rightarrow (a \in A \leftrightarrow b \in A)$: the information in g_a^S determines membership or non-membership of any $a \in Y$ in A.

We will say that a set A is (X, Y)-symmetric iff it is (X, Y)-symmetric with support some finite subset S of Y.

Lemma: Let X be a set and let Y be a subset of $\mathcal{P}(X)$ which is closed under finite intersections and complements relative to X. All finite subsets of Y are (X, Y)-symmetric. The relative complement with respect to Y of an (X, Y)-symmetric set is (X, Y)-symmetric. Any finite intersection of (X, Y)-symmetric sets is (X, Y)-symmetric.

Proof of Lemma: A finite subset S of Y is (X, Y)-symmetric with support S itself: an element of S is clearly uniquely identified by its cardinalities relative to the elements of $V(S)$. Suppose that $S \subseteq Y$ is finite, $A \subseteq Y$, and membership of any $y \in Y$ in A is decided by the cardinalities of y relative to the elements of $V(S)$. Then non-membership of any $y \in Y$ in A is also so decided, so $Y - A$ is (X, Y)-symmetric. Suppose that A is (X, Y)-symmetric with support S and B is (X, Y)-symmetric with support T. Observe that if we know the relative cardinalities of a set $y \in Y$ with respect to the elements of $V(S \cup T)$ then we know its relative cardinalities with respect to the elements of $V(S)$ and the elements of $V(T)$ [the relative cardinality of y with respect to $s \in V(S)$ (for example) is the sum of the relative cardinalities of y with respect to sets $u \in V(S \cup T)$ such that $u \subseteq s$]: thus membership of $y \in Y$ in either A or B is decided by the relative cardinalities of y with respect to the elements of $V(S \cup T)$, and membership of $y \in Y$ in $A \cap B$ (or $A \cup B$) is also so decided: $A \cap B$ (and $A \cup B$) are (X, Y)-symmetric with support $S \cup T$.

Observation: It is useful to note that if a set A is (X, Y)-symmetric with support S, it will be the case that for any permutation f of X such that $f_2(Y) = Y$ (basically a 1-setlikeness condition) and such that f_1 fixes each element of S we have $f_2(A) = A$. The converse will be true if the additional condition holds that for every $a, b \in Y$ such that $|a| = |b|$, there is a permutation g of X such that $g_2(Y) = Y$ and $g_1(a) = b$. This observation is intended to clarifiy the relevance of this condition to the axiom we are trying to model.

Definition: Let X be a set of cardinality $\mathbf{c}$. A set Y is a *pseudo-power set* of X iff $Y \subseteq \mathcal{P}(X)$, Y contains all finite subsets of X, Y is closed under complement relative to X and under finite intersections, each infinite element of Y is of cardinality $\mathbf{c}$ and each infinite element of Y is the union of two disjoint infinite elements of Y.

Example: Let X be the set of sequences of natural numbers. For each finite sequence s of natural numbers let I_s be the set of sequences of natural numbers with s as an initial segment. Let Y_1 be the smallest set containing all I_s's and closed under relative complement with respect to X and finite intersections. Let Y be the collection of all subsets of X with finite symmetric difference from an element of Y_1. Y is a pseudo-power set of X.

Definition: A *type sequence* is a sequence of sets $\{X_i\}_{i \in \mathbb{N}}$ with the following properties:

1. X_0 is of cardinality $\mathbf{c}$.
2. X_1 is a pseudo-power set of X_0.
3. For each $i \in \mathbb{N}$, X_{i+2} is the set of all (X_i, X_{i+1})-symmetric subsets of X_{i+1}.

Clearly sets X_0 and X_1 can be chosen which satisfy the stated conditions, and once X_0 and X_1 are chosen the entire type sequence is determined.

We fix a type sequence X_i in what follows. We prove a sequence of lemmas about this fixed type sequence (and so about all type sequences).

Lemma: X_{i+1} contains all finite subsets of X_i and is closed under complement relative to X_i and finite intersection.

Proof of Lemma: For $i = 0$ this follows from X_1 being a pseudo-power set of X_0. For $i \geq 1$, this follows from the same closure properties holding of the collection of (X_{i-1}, X_i)-symmetric sets (which is of course X_{i+1}).

We define a basic notion which is important for our argument.

Definition: We call a cardinal *relevant* if it is either finite or $\mathbf{c}$. We call a relative cardinal relevant if both its components are relevant. We will establish that all sets belonging to any term of a type sequence have relevant cardinality.

Definition: We call a function $g : V(S) \to \texttt{Card}^2$ a formal element of X_{i+2} (over S) if S is a finite subset of X_{i+1}, and each element of the range of g is a relevant relative cardinal, and for each $T \in V(S)$ we have $\pi_1(g(T)) + \pi_2(g(T)) = |T|$.

We make a series of observations about formal elements.

formal elements determined by elements: It should be obvious that for each finite subset S of X_{i+1} and $a \in X_{i+1}$, g_a^S is a formal element of X_{i+2} over S, as long as all cardinalities of elements of X_{i+1} are relevant.

sets in X_{i+2} are unions of formal elements: Assume that all elements of X_{i+1} are of relevant cardinality: it follows that the elements of X_{i+2} are exactly the sets which are for some finite $S \subseteq X_{i+1}$ unions (possibly infinite) of sets of the form $\{a \mid g_a^S = g\}$ where g is a formal element. Each element of X_{i+2} with support S is the union of the set of all sets of the form $\{a \mid g_a^S = g\}$ which meet it.

stronger observation would follow from a splitting property: Further (again under the assumption that each element of X_{i+1} is of relevant cardinality), we can state that each set B of formal elements of X_{i+2} with a common domain $V(S)$ determines a unique element of X_{i+2} ($\bigcup\{\{a \mid g_a^S = g\} \mid g \in B\}$) if we can establish that no set $\{a \mid g_a^S = g\}$ is empty for g a formal element with domain $V(S)$. This will be true if X_{i+1} has the following general splitting property: if $a \in X_{i+1}$ has cardinality $\mu = \kappa + \lambda$, where κ and λ are relevant cardinals, then there are sets b and c in X_{i+1} which are disjoint, have union a, and have cardinalities κ, λ respectively. Then, given a formal element g, we can select sets which have the correct cardinalities relative to each $T \in V(S)$ [sets a_T with $g(T) = |a_T|_T = (|a_T \cap T|, |T - a_T|)$] and take finite unions (under which X_{i+1} is closed) to obtain a set $a = \bigcup_{T \in V(S)} a_T$ with $g = g_a^S$. We already know that X_1 has the indicated splitting property; we will show by induction that all X_{i+1} have this property.

We now prove a lemma by induction on the X_i's:

Cardinality and Splitting Property Lemma: Each set in each X_{i+1} is either finite or of cardinality $\mathbf{c}$, $|X_i| = |X_{i+1}| = \mathbf{c}$ for each i, and each X_{i+1} has the splitting property.

Proof of Lemma: We prove this by induction.

The case $i = 0$ is immediate from the fact that X_1 is a pseudo-power set of X_0. Assume this is true for $i = k$ and show that it is true for $i = k + 1$.

We first consider the cardinalities of sets $\{a \in X_{k+1} \mid g^S_a = g\}$. Fix a finite subset S of X_{k+1} and a formal element g of X_{k+1} over S.

We claim that $|\{a \in X_{k+1} \mid g^S_a = g\}|$ is the cardinality of the set of functions h with domain $V(S)$ such that for each $T \in V(S)$ we have $h(T) \subseteq T$, $h(T) \in X_{k+1}$, and $|h(T)|_T = g(T)$. We indicate the one-to-one correspondence: for any a with $g^S_a = g$ the associated h is $(T \mapsto a \cap T)$ and for any h with the indicated property $a = \bigcup_{T \in V(S)} h(T)$ is the corresponding a such that $g^S_a = g$.

This implies that

$$|\{a \in X_{k+1} \mid g^S_a = g\}| = \prod_{T \in V(S)} |\{A \in X_{k+1} \cap \mathcal{P}(T) || A|_T = g(T)\}|$$

We abbreviate $|\{A \in X_{k+1} \cap \mathcal{P}(T) || A|_T = g(T)\}|$ as $\kappa_{g,T}$. We will see that $\kappa_{g,T}$ is determined by the relative cardinal $g(T)$.

If $|T|$ is finite, $g(T) = (m, n)$ for natural numbers m and n and $\kappa_{g,T} = \begin{pmatrix} m+n \\ n \end{pmatrix}$, a binomial coefficient. Recall that all finite subsets of X_{k+1} and so all subsets of T belong to X_{k+1}.

Otherwise $|T| = \mathbf{c}$ by inductive hypothesis. If $g(T) = (\mathbf{c}, 0)$ or $(0, \mathbf{c})$, obviously $\kappa_{g,T} = 1$.

If $g(T) = (\mathbf{c}, n)$ or $(n, \mathbf{c})$, for n finite and nonzero, then $\kappa_{g,T} = \mathbf{c}$: every finite or cofinite subset of T belongs to X_{k+1}, and there are $\mathbf{c}$ subsets of T of each nonzero finite cardinality and $\mathbf{c}$ complements of these sets. So $\kappa_{g,T} = \mathbf{c}$.

If $g(T) = (\mathbf{c}, \mathbf{c})$ (the only other alternative, again by inductive hypothesis), then there is at least one way to split the set A into two subsets B, C of size $\mathbf{c}$, by the splitting property for X_{k+1} assumed in the inductive hypothesis. There cannot be more than $\mathbf{c}$ ways to do this, because there are $\mathbf{c}$ sets in X_{k+1}; to see that there are at least $\mathbf{c}$ different ways, consider that A can be partitioned into $B - \{b\}$ and $C \cup \{b\}$ for each of the $\mathbf{c}$ different elements b of B. It follows that $\kappa_{g,T} = \mathbf{c}$.

We now observe that the cardinality of $\{a \in X_{k+1} \mid g_a^S = g\}$ is the product of finitely many cardinals each of which is either finite or $\mathbf{c}$, and so is either finite or $\mathbf{c}$. We further observe that this cardinality is determined entirely by the range of g (the set of relevant relative cardinals $g(T)$ for $T \in V(S)$): the way in which it is computed has just been given.

Further, observe that there are exactly ω formal elements over any S (there are ω possible pairs of cardinals which can be images, because there are ω possible cardinalities for subsets of X_{i+1}, the finite cardinalities and $\mathbf{c}$, and at least one of the elements of $V(S)$ is infinite and so actually admits ω possible images).

Each set in X_{k+2} is the union of a possibly infinite collection of sets $\{a \in X_{k+1} \mid g_a^S = g\}$ for a fixed S. For each S, there are no more than ω sets $\{a \in X_{k+1} \mid g_a^S = g\}$ (because there are no more than ω formal elements g over S) and so no more than $2^\omega = \mathbf{c}$ such unions, and there are $\mathbf{c}$ finite subsets S of X_{k+1}, so there are no more than $\mathbf{c}$ elements of X_{k+2}. Since each of the $\mathbf{c}$ singletons of elements of X_{k+1} belongs to X_{k+2}, the cardinality of X_{k+2} is exactly $\mathbf{c}$.

Only finitely many of the sets $\{a \in X_{k+1} \mid g_a^S = g\}$ for a fixed S can be finite: for $|\{a \in X_{k+1} \mid g_a^S = g\}|$ to be finite, g must map each infinite T in $V(S)$ to either $(0, \mathbf{c})$ or $(\mathbf{c}, 0)$, and of course there are only finitely many possible images for g at a finite T, so only finitely many functions g will give a finite cardinality for $\{a \in X_{k+1} \mid g_a^S = g\}$. The cardinality of any element of X_{k+2} is the sum of finitely many or countably many cardinals, each either finite or $\mathbf{c}$, no more than finitely many of which can be finite, so is either finite or $\mathbf{c}$.

It remains to show that X_{k+2} has the splitting property. Any infinite element A of X_{k+2} has an infinite subset, also in X_{k+2}, of the form $\{a \in X_{k+1} \mid g_a^S = g\}$.

Suppose S is nonempty. The formal element g must split some infinite element B of $V(S)$ nontrivially (map it to something other than $(0, \mathbf{c})$ or $(\mathbf{c}, 0)$.) By inductive hypothesis, the set B is the union of disjoint sets C and D, both of cardinality $\mathbf{c}$, belonging to X_{k+1}. Now consider the formal element g_1 mapping each element of $V(S \cup \{C, D\})$ other than C or D to its image under g, C to $g(B)$, and D to $(0, \mathbf{c})$ if $\pi_2(g(B))$ is finite and to $(\mathbf{c}, 0)$ otherwise, and g_2 defined in the same way but interchanging the roles of C and D. The sets $\{a \in X_{k+1} \mid g_a^S = g_1\}$ and $\{a \in X_{k+1} \mid g_a^S = g_2\}$ are disjoint subsets, both of cardinality $\mathbf{c}$, of the set $\{a \in X_{k+1} \mid g_a^S = g\}$ and so of A, and of course they belong to X_{k+2}; the fact that X_{k+2} is closed under finite intersections and relative complements (and so under finite unions) implies that if a set in X_{k+2} has two disjoint subsets in X_{k+2} of cardinality $\mathbf{c}$ it is the union of two disjoint sets of cardinality $\mathbf{c}$ belonging to X_{k+2}.

Suppose S is empty. Then the map g sends the sole element X_{k+1} of its domain to a pair of cardinals of one of the forms $(n, \mathbf{c})$, $(\mathbf{c}, n)$, or $(\mathbf{c}, \mathbf{c})$. X_{k+1} itself can be split into infinite disjoint sets $C, D \in X_{k+1}$ by inductive hypothesis. For E equal to either C or D, sonsider the set G_E in X_{k+2} with support $\{E\}$ which maps E to $g(X_{k+1})$ and E^c to $(0, \mathbf{c})$ if $\pi_2(g(X_{k+1})) = \mathbf{c}$ and to $(\mathbf{c}, 0)$ otherwise. The sets G_C and G_D are disjoint sets of cardinality $\mathbf{c}$ which are both subsets of A and belong to X_{k+2}, so we are done.

Corollary of the Lemma: X_{i+1} is a pseudo-power set of X_i for each i (this is immediate). It follows that each tail of a type sequence is also a type sequence.

We now extract a sort of common core of all type sequences.

Definition: Let $\{X_i\}$ be a type sequence. We define X_1^* as $\emptyset$. When $X_i^* \subseteq X_i$ has been defined, we define X_{i+1}^* as the set of all $A \in X_{i+1}$ with support $S \subseteq X_i^*$. Note that X_2^* is nonempty, as there are sets with empty support, such as X_1 itself. We refer to the elements of the X_i^* as "computable" sets relative to this type sequence.

Definition: Let $\{X_i\}$ and $\{Y_i\}$ be type sequences. For each $i \geq 1$, we define a map $\sigma_i : X_i^* \to Y_i^*$. The definition of σ_1 presents no difficulties as it is the empty map.

We define σ_{i+1} once σ_i has been defined with the following properties: σ_i is injective, for each $A \in X_i^*$, $|\sigma_i(A)| = |A|$, for each $A \in X_i^*$, $\sigma_i(X_{i-1} - A) = Y_{i-1} - \sigma_i(A)$, and for each $A, B \in X_i^*$, $\sigma_i(A \cap B) = \sigma_i(A) \cap \sigma_i(B)$. Note that σ_1 trivially has each of these properties.

Suppose that σ_i has been defined with the correct properties. Let A be an element of X_{i+1}^*. It has a support $S \subseteq X_i^*$ because it is computable. Each element T of $V(S)$ is an intersection of some elements of S and some complements relative to X_i of elements of S: by the assumed properties of σ_i, $\sigma_i(T)$ is an intersection of some elements of σ_i"S and some complements relative to Y_i of elements of σ_i"S, that is an element of $V(\sigma_i$"$S)$. Further, we can identify an element of $V(S)$ which is the preimage under σ_i of a given element of $V(\sigma_i$"$S)$ in the same way. Thus σ_i restricted to $V(S)$ is a bijection onto $V(\sigma_i$"$S)$. Further, σ_i preserves cardinality by assumed properties of σ_i, so a map g is a formal element over S iff $g \circ \sigma^{-1}$ is a formal element over σ"S. We denote $g \circ \sigma_i^{-1}$ by g^{σ_i}.

We claim that we can define $\sigma_{i+1}(A)$ as the set with support σ_i"S containing all a' such that $g_{a'}^{\sigma_i\text{“}S} = (g_a^S)^{\sigma_i}$ for some $a \in A$.

To verify that this is a definition, we need to verify that if S and T are both supports of A, the set with support σ_i"S containing all a' such that $g_{a'}^{\sigma_i\text{“}S} = (g_a^S)^{\sigma_i}$ for some $a \in A$ is the same set as the set with support σ_i"T containing all a' such that $g_{a'}^{\sigma_i\text{“}T} = (g_a^T)^{\sigma_i}$ for some $a \in A$. Suppose that we have a' such that $g_{a'}^{\sigma_i\text{“}S} = (g_a^S)^{\sigma_i}$ for some $a \in A$. Compute $g_{a'}^{\sigma_i\text{“}(S\cup T)}$. This is h^{σ_i} for some formal element h over $S \cup T$. Choose any a^* such that $g_{a^*}^{S\cup T} = h$. From $g_{a^*}^{S\cup T}$ we can compute both $g_{a^*}^S$ – for which we must have $g_{a^*}^S \circ \sigma_i^{-1} = g_{a'}^{\sigma_i\text{“}S} = g_a^S \circ \sigma_i^{-1}$, for some $a \in A$, so $a^* \in A$ as well, and also we can compute $g_{a^*}^T$, for which we must have $g_{a^*}^T \circ \sigma_i^{-1} = g_{a'}^{\sigma_i\text{“}T}$, so we see that $g_{a'}^{\sigma_i\text{“}T} = (g_a^T)^{\sigma_i}$ for some $a \in A$ (namely the a^* just selected). This shows an inclusion in one direction between the two sets, but the situation is symmetrical, so the sets are equal.

That $\sigma_{i+1}(A)$ has the same cardinality as A follows from the general fact that $|\{a \mid g_a^S = g\}|$ is computable solely from the range of the formal element g (in a way which does not depend at all on what type sequence one is working with). The formal elements over σ_i"S associated with elements of $\sigma_{i+1}(A)$ correspond precisely to the formal elements over S associated with elements of A, so the sets are the same size. This is also the reason that σ_{i+1} is injective: it could only fail to be injective if some formal element represented in A failed to be represented in $\sigma_{i+1}(A)$, and we have seen above that this does not happen.

That $\sigma_{i+1}(A \cup B) = \sigma_{i+1}(A) \cup \sigma_{i+1}(B)$ and $\sigma_{i+1}(A - B) = \sigma_{i+1}(A) - \sigma_{i+1}(B)$ (the latter being of particular interest when $A = X_i$) is readily seen by considering the definitions of the sets involved using the same support for A and B (the union of a support for A and a support for B will do).

This completes the verifications of the properties of σ_{i+1} needed for the induction to continue. Although it is not a fact used in the definition, it should be clear that each σ_i is also onto Y_i^* (a quick way to see this is to consider the exactly symmetric definition of the inverse of σ_i).

Observation: Let $\{X_i\}$ and $\{Y_i\}$ be type sequences and let the maps σ_i be defined as above. If $A \in X_i^*$ and $B \in X_{i+1}^*$ and $A \in B$, it follows that $\sigma_i(A) \in \sigma_{i+1}(B)$. (The implication is actually a biconditional, by considering the inverse map.)

Proof: Membership of A in B depends entirely on the sizes of the intersections of A and $X_{i-1} - A$ with elements of $V(T)$ for a support T of B. Membership of $\sigma_i(A)$ in $\sigma_{i+1}(B)$ depends entirely on the sizes of the intersections of $\sigma_i(A)$ and $Y_{i-1} - \sigma_i(A)$ with elements of $V(\sigma_i``T)$, as $\sigma_i``T$ is known to be a support for $\sigma_{i+1}(B)$. Corresponding sizes here will be the same by the known properties of σ_i, so we will obtain the same truth values for the two membership facts.

This shows that in effect all type sequences have isomorphic "cores" of computable sets.

Observation: For any type sequence $\{X_i\}$, the sequence $\{Y_i\}$ defined by $Y_i = X_{i+1}$ is also a type sequence: we noted above that tails of type sequences are also type sequences. Further, $Y_i^* \subseteq X_{i+1}^*$ for each i: this is obvious for $i = 1$ as $Y_1 = \emptyset$; Y_{i+1}^* is inhabited by sets in $X_{i+2} = Y_{i+1}$ with support taken from Y_i^*, which is a subset of X_{i+1}^* by inductive hypothesis, so we see that such sets are also in X_{i+2}^*.

Construction: Fix a type sequence $\{X_i\}$ in what follows. Let $Y_i = X_{i+1}$ as in the previous observation and let the maps $\sigma_i : X_i^* \to Y_i^*$ be defined as above, noting that in this situation $\sigma_i : X_i^* \to X_{i+1}^*$ is injective but not expected to be onto X_{i+1}^*. The construction that follows depends in no essential way on the choice of the specific type sequence.

We now define a structure (M, ϵ) which will turn out to be the model of *NFMSI* at which we aim.

Definition: We define M as the set of all sequences s with domain a final segment of $\mathbb{Z}^+$ with the property that $s_i \in X_i^*$ for each i in the domain of s, $s_{i+1} = \sigma_i(s_i)$ for all values of i in the domain of s, and for any i, if $\sigma_i^{-1}(s_{i+1})$ exists then $i \in \mathtt{dom}(s)$ and $s_i = \sigma_i^{-1}(s_{i+1})$. Note that each element of any X_i is a term of exactly one sequence in M. For two elements $s, t \in M$, we define $s \,\epsilon\, t$ as holding iff $s_i \in t_{i+1}$ for some appropriate i (and so for all appropriate i).

Definition (support of an element of M): Let $t \in M$ have domain $[k, \infty)$. We say that a finite set $S \subseteq M$ is a support for t iff s_{k-1} is defined for each $s \in S$ and $\{s_{k-1} \mid s \in S\}$ is a support for t_k. Of course there is a support for t because there is a support for t_k with computable elements. For any i in the domain of t, $\{s_{i-1} \mid s \in S\}$ will be a support for t_i. There must be an $s \in S$ such that s_{k-2} is not defined, or $\{s_{k-2} \mid s \in S\}$ would give a support for $\sigma_{k-1}^{-1}(t_k)$, which does not exist because $k - 1 \notin \mathtt{dom}(t)$. Note that induction on support is thus an appropriate method for proofs about M, since the index of the first element of a sequence s in M strictly dominates the indices of the first elements of the sequences in a support of s.

Lemma: For each $s \in M$ and each i in the domain of s, $|s_i| = |\{t \mid t \epsilon s\}|$. That is, the cardinality of the inverse image under ϵ of an element of M is the same as the actual cardinality of each of its terms.

Proof of Lemma: For any sequence $s \in M$, each s_i has the same cardinality (because σ_i's preserve cardinals). First we show that for any s with a finite ϵ-preimage. each s_i has the same cardinality as that ϵ-preimage. It is sufficient to prove that every finite computable set has only computable elements. A finite computable set has a support with computable elements. Elements of the set must be unions of infinite elements of the Venn diagram of the support (which are computable) [infinite elements of the Venn diagram will not be split because the set would then not be finite] and finite subsets of finite elements of the Venn diagram of the support (which are computable because the computable finite compartments of the Venn diagram have computable elements by inductive hypothesis, and finite sets of computable sets are computable). It is then clear that the elements of a finite computable set are computable, and it follows that for any s_i, the elements of s_i are each of the form t_{i-1} for some $t \epsilon s$; all $t \epsilon s$ must correspond to some $t_{i-1} \in s_i$ because all the sets s_i are of the same finite size.

The infinite case is more complicated. We know that it must be. All of the computable sets in X_2, for example have no computable elements at all, whereas the analogous sets in higher X_i's have infinitely many computable elements.

We define the Frege natural number n_{i+2} as the set of all n-element subsets of X_i, which is clearly an element of X_{i+2} with empty support. We define the Frege co-natural n^*_{i+2} as the set of all $A \in X_{i+1}$ such that $|X_i - A| = n$; again, this is obviously in X_{i+2} with empty support. We define Σ_{i+2} as the collection of all elements A of X_{i+1} such that A and $X_i - A$ are both infinite (and so of cardinality $\mathbf{c}$). Note that there are $\mathbf{c}$ distinct unions of Frege naturals n_{i+2}, all of which are elements of X_{i+2} with empty support. The elements of X_{i+2} with empty support are exactly the sets of the forms $Z^{A,B}_{i+2} = \bigcup\{n_{i+2} \mid n \in A\} \cup \bigcup\{n^*_{i+2} \mid n \in B\}$ and $Z^{A,B}_{i+2} \cup \Sigma_{i+2} = \bigcup\{n_{i+2} \mid n \in A\} \cup \bigcup\{n^*_{i+2} \mid n \in B\} \cup \Sigma_{i+2}$, where A and B are sets of natural numbers. The action of the maps σ on such sets is simple: $\sigma_{i+2}(Z^{A,B}_{i+2}) = Z^{A,B}_{i+3}$ and $\sigma_{i+2}(Z^{A,B}_{i+2} \cup \Sigma_{i+2}) = Z^{A,B}_{i+3} \cup \Sigma_{i+3}$

We show that for any $s \in M$, for all large enough i, there are as many computable elements of s_i as there are elements of s_i.

We have already seen that this is true if any s_i is finite.

We prove this for infinite sets by induction on support.

Suppose that s is a sequence in M whose terms are infinite sets and have empty support. For any such sequence s, either Σ_{i+2} will be a subset of every s_{i+2} or there is an $n \in \mathbb{N}$ such that either n_{i+2} is a subset of s_{i+2} for each i or n^*_{i+2} is a subset of s_{i+2} for each i, by the analysis of sets of empty support given above. So it is enough to show that n_{i+2} has $\mathbf{c}$ computable elements for large enough i, n^*_{i+2} has $\mathbf{c}$ computable elements for large enough i, and Σ_{i+2} has $\mathbf{c}$ computable elements for large enough i.

X_2 has $\mathbf{c}$ distinct computable elements (arbitrary unions of the Frege naturals n_2 work); finite sets of these are computable elements of X_3, and there are $\mathbf{c}$ of these finite sets of each cardinality n, each of which will belong to n_4 which is thus seen to have $\mathbf{c}$ computable elements. It is then obvious that each n_i for $i > 4$ has $\mathbf{c}$ computable elements.

X_2 has **c** distinct computable elements (arbitrary unions of the Frege naturals n_2 work); finite sets of these of a fixed cardinality n are computable elements of X_3, and there are **c** relative complements of these finite sets with respect to X_2, each of which will belong to n_4^* which is thus seen to have **c** computable elements. It is then obvious that each n_i^* for $i > 4$ has **c** computable elements.

There are **c** singletons of computable elements of X_2, all of which belong to 1_4. There are **c** computable non-singletons belonging to $X_3 - 1_4$ (such as sets of two computable elements of X_2). Thus $1_4 \in \Sigma_5$, and also for each of the **c** computable elements x of X_2, $1_4 - \{x\} \in \Sigma_5$. Each of the sets $1_4 - \{x\}$ is computable for x computable, so Σ_5 (and each Σ_i for $i > 5$) has **c** elements.

This completes the basis step.

Choose a k large enough that for each element t of a support of s (derived as described above from a support of the first term of s), t_k has as many computable elements as it has elements (the existence of such a k is immediate from the inductive hypothesis). Elements of s_{k+1} are constructed from the t_k's by choosing appropriate numbers of elements from each t_k and taking a union. If we are choosing finitely many elements to include or exclude from each t_k, we can clearly do this in a way which gives as many computable elements as elements. The only difficult case is the split into two subsets of size **c**. In fact, all we need is to show that any infinite computable set can be split into two infinite computable sets: once we have one split of an appropriate t_k into two infinite computable parts, we can move a single computable element from one part to the other to get **c** distinct splits.

What remains is to show that an infinite computable set can be split into two infinite computable sets. The argument proceeds exactly as in the proof of the splitting property in the proof of the Cardinality and Splitting Property Lemma above with one additional observation. The infinite set must have a support with computable elements. Some infinite element of this support must be nontrivially split, and this infinite element can be split into two infinite computable sets by inductive hypothesis. In the case where the support is nonempty, it is clear that the sets constructed will be computable. In the case where the support is empty, one has to appeal to the fact that X_i can be split into two infinite computable sets for i large enough: we can split X_i into 1_{i+1} and $X_i - 1_{i+1}$, which are computable and infinite.

Definition: We denote the inverse image of $s \in M$ under ϵ by $E(s)$; that is, $E(s) = \{t \in M \mid t \,\epsilon\, s\}$.

Lemma: (M, ϵ) is a structure for the language of set theory with the property that each element s of M is uniquely determined by its inverse image $E(s)$ under ϵ and the subsets of M which are of the form $E(s)$ for some $s \in M$ are exactly those sets $A \subseteq M$ for which there is a finite collection $S \subseteq M$, such that, defining $\Sigma = E``S = \{E(t) \mid t \in S\}$, we have that for all $a, b \in M$, it is the case that $a \in A \leftrightarrow b \in A$ if for every $T \in V(\Sigma)$ we have $|E(a) \cap T| = |E(b) \cap T|$ and $|T - E(a)| = |T - E(b)|$ (that is, $a \in A \leftrightarrow b \in A$ if $(\forall T \in V(\Sigma)).|E(a)|_T = |E(b)|_T)$).

Proof of Lemma: Suppose $s, t \in M$ and that $s \neq t$. Then $s_k \neq t_k$ for some k. Thus $s_k \Delta t_k$ is a computable nonempty set, and there is $v \in M$ such that $v_k = s_k \Delta t_k$. Since v_k is nonempty, the ϵ-preimage of v is nonempty (since it has the same cardinality), whence we have $u \in M$ such that for some n, $u_{n-1} \in v_n = s_n \Delta t_n$, whence $u \,\epsilon\, s$ and $u \,\epsilon\, t$ have different truth values. So we have shown that distinct elements of M have distinct inverse images under ϵ.

Suppose $A \subseteq M$ and $A = E(s)$ for some $s \in M$. Let S be a support for s. Suppose for $a, b \in M$ we have $|E(a) \cap T| = |E(b) \cap T|$ and $|T - E(a)| = |T - E(b)|$ for every $T \in V(E``S)$. We know that for any k in the domain of s, $S_{k-1} = \{t_{k-1} \mid t \in S\}$ is a support for s_k. Further, we have $|a_{k-1} \cap T| = |b_{k-1} \cap T|$ and $|T - a_{k-1}| = |T - b_{k-1}|$ for every $T \in V(S_{k-1})$ as long as a_{k-1} and b_{k-1} are defined [because we know that $V(S_{k-1})$ and $V(E``S)$ are precisely parallel in structure in a suitable sense], whence we have $a_{k-1} \in s_k$ iff $b_{k-1} \in s_k$, whence $a \,\epsilon\, s \leftrightarrow b \,\epsilon\, s$, whence $a \in A \leftrightarrow b \in A$.

Suppose we are given $A \subseteq M$ for which there is a finite collection $S \subseteq M$, such that, defining $\Sigma = E``S = \{E(t) \mid t \in S\}$, we have that for all $a, b \in M$, it is the case that $a \in A \leftrightarrow b \in A$ if for every $T \in V(\Sigma)$ we have $|E(a) \cap T| = |E(b) \cap T|$ and $|T - E(a)| = |T - E(b)|$. Choose k the least index such that t_{k-1} is defined for each $t \in S$. Define S_{k-1} as above. Each element of S_{k-1}, which we may denote by u, determines a unique $U \in M$ such that $U_{k-1} = u$, which in turn determines an element

$E(U)$ of $V(\Sigma)$ which we will denote by u^*. Note that $|u| = |u^*|$. Define σ as the set of all $c \in X_{k-1}$ such that there is $c^* \in A$ such that for each $u \in S_{k-1}$ we have $|c \cap u| = |E(c^*) \cap u^*|$ and $|u - c| = |u^* - E(c^*)|$. Clearly $\sigma \in X_k^*$. Then the unique element s of M such that $s_k = \sigma$ will satisfy $E(s) = A$.

Discussion: Our intention here is to "identify" each element x of X_i^* with its "analogue" $\sigma_i(x)$ in X_{i+1}^* (and to discard all the elements of each $X_i - X_i^*$). We know that the maps σ_i distribute over intersection and relative complement and that for $x, y \in X_i^*, X_{i+1}^*$ respectively, we have $x \in y \leftrightarrow \sigma_i(x) \in \sigma_{i+1}(y)$, so membership and boolean operations on sets have coherent definitions after this identification. We know that σ_i preserves cardinality of $x \in X_i^*$, but that is the number of elements x has in X_{i-1}, not in X_{i-1}^*: but we have shown above that some analogue of x in a higher X_j will have as many elements in X_{j-1}^* as it has in X_{j-1}; so the notion of cardinality is also preserved in the structure M obtained from the identification, in spite of the fact that elements of $X_i - X_i^*$ are discarded (and this also ensures that extensionality holds in M). Finally, this means that if $x \in X_i$ has a support $S \subseteq X_{i-1}$, it follows that after the identification it has in effect the same support, because we have shown that all the constituent notions are preserved. After the identification, we obtain a structure in which the "membership relation" is non-well-founded, because for example the sequence $V_i = X_{i-1}$ has as its preimage under ϵ the whole of M (including itself).

Definition (extending structure to add classes): Observe that no subset of M is an element of M in the usual set theoretical sense. This makes it very convenient to extend (M, ϵ) to the structure $(M^+, \epsilon^+) = (M \cup \mathcal{P}^*(M), \epsilon \cup (\in \cap(M \times \mathcal{P}^*(M))))$ where $\mathcal{P}^*(M)$ is the set of all subsets of M which are not preimages under ϵ. We claim that this is a model of the theory of sets and classes with $k = 1$ which is our target.

After this point we speak the language of M^+, considered as a structure for the language of set theory: we allow ourselves to use $\in$ to represent ϵ^+ and speak of elements of M^+ as if they had the elements of their ϵ^+ preimages as their elements. We speak of elements of M^+ as classes and elements of M as sets; we say that an element of M^+ has cardinality κ iff its preimage under ϵ^+ has cardinality κ. The concepts of support of an element of M, formal element of an element of M over a support. and formal element g_a^S associated with $a \in M$ where S is a finite subset of M all make sense here. Our aim is to show that (M^+, ϵ^+) is a model of our symmetric theory of sets and classes with $k = 1$.

In the discussion below, setlike permutations are defined as in the first parts of the paper, relative to the notions of set and class in the structure (M^+, ϵ^+).

Articulation of basic claim: It is sufficient to show that in the theory of the extended structure, for any sets A and B both of cardinality $\mathbf{c}$ (in the external sense) there is a setlike bijection from A to B. We indicate why this is sufficient.

Demonstration that the basic claim is sufficient: Suppose that A is a set (an element of M in the usual sense). Then there is a finite set S such that for any setlike permutation f such that $f_1(s) = s$ for each $s \in S$, we have $f_2(A) = A$. The usual support of A will work: the setlike permutations f with the indicated property preserve cardinality and map sets to sets, and this is enough to ensure that for any $a \in A$, $f_1(a)$ will be a set and we will have $|a \cap s| = |f_1(a) \cap s|$ and $|a^c \cap s| = |f_1(a)^c \cap s|$ for each $s \in V(S)$, so $a \in A \leftrightarrow f_1(a) \in A$.

Now suppose that A is a class and S is a finite subset of M with the property that for any setlike permutation f such that $f_1(s) = s$ for each $s \in S$, $f_2(A) = A$. We want to show that if a and b are sets with the same cardinalities relative to the elements of the Venn diagram over S, then either both belong to A or both do not belong to A: this would be enough to show that A is a set. What we need is a setlike permutation f with f_1 fixing each element of S such that $f_1(a) = b$. We construct it by pasting together setlike bijections between $a \cap T$ and $b \cap T$ and between $T - a$ and $T - b$ for each $T \in V(S)$. For finite sets these obviously exist; the basic claim is enough to show that the component bijections always exist; it is clear that we can take their union and obtain a setlike bijection with the desired properties.

Further simplification of basic claim: It should be clear that it is sufficient to show that for any infinite set A there is a setlike bijection from V onto A; we can then use composition to construct a setlike bijection from any infinite A to any other infinite B.

Showing that the desired property holds is a bit tricky, and involves exploiting symmetry properties of M.

Lemma (all sets are symmetric): For every set A there is a natural number n such that $f_n(A) = A$ for any setlike permutation f of the universe at all. We prove this by induction on support. For sets with empty support, $n = 2$ works. For a set A with nonempty support, let B be the element of the support with the largest associated natural number m such that $f_m(B) = B$ for any setlike permutation f, and it follows that $f_{m+1}(A) = A$ for any setlike permutation f.

Definition (convenient notation for singleton-related operations): We introduce the notation $\iota(A)$ for $\{A\}$. The point of this notation is that it can be iterated: define $\iota^0(A)$ as A and $\iota^{k+1}(A)$ as $\{\iota^k(A)\}$. For any set A, we define ι"A as the class of singletons of elements of A, which is the set determined by the formal element with domain $\{A, A^c\}$ sending A to $(1, |A| - 1)$ and A^c to $(0, |A^c|)$ (where $A \neq V$; the case $A = V$ is simpler).

Theorem (stronger version of basic claim): For any infinite set A there is a setlike injection f from V into A, with the additional property that there is k such that for all setlike bijections g, $f(g(x)) = g_k(f(x))$.

Proof of Theorem: If A has empty support, it is sufficient to consider the case where A is the set associated with a single formal element (that is, the collection of all sets with cardinality κ whose complements have cardinality λ). We can assume $\kappa\lambda \neq 0$, as otherwise the set in question is finite.

Choose a fixed set B of $\kappa - 1$ non-singletons, and map each $a \in A$ to $F(a) = \{\{\iota^p(a)\}\} \cup B$, where p is chosen large enough so that B is $(p+2)$-symmetric. That F is a bijection is evident. That F"C is a set for any set C can be seen by describing F"C as the set of all sets E such that E contains one element of ι^{p+1}"C, E contains $|B|$ elements of B, $V - E$ contains 0 elements of B, and E contains 0 elements of $V - B - \iota^{p+1}$"C. The description given of F"C should make it clear that it is symmetric with support $\{B, \iota^{p+1}$"$C\}$ and thus a set, so F is setlike. Note further that for any setlike bijection g, $F(g(x)) = \{\{\iota^p(g(x))\}\} \cup B = g_{p+2}(\{\{\iota^p(x)\}\} \cup B) = g_{p+2}(F(x))$, recalling that B is $(p+2)$-symmetric. It should be evident that $F(a) \in A$ for any a.

If A has nonempty support, then the Venn diagram of its support contains at least one infinite element T which is nontrivially split by at least one formal element of A. We will construct a setlike bijection from V into the subset of A associated with that formal element. Choose $B \in A$ with g^S_B the desired formal element. One of the sets $B \cap T$ and $T - B$ has infinite cardinality. Call it D. By inductive hypothesis there is a setlike bijection G from V into D, and k such that $G(f(x)) = f_k(G(x))$ for any setlike permutation f. Fix an element x of $T - D$. Define $F(a)$ as $B - \{x\} \cup \{G(\iota^p(a))\}$ or $B \cup \{x\} - \{G(\iota^p(a))\}$ depending on whether $x \in B$ or $x \notin B$, where p is chosen so that B and $\{x\}$ are $(p+k+1)$-symmetric. Clearly $F(a) \in A$ for each $a \in V$ and F is injective. For any set C, $F``C$ is the set of all sets E such that E contains 0 elements of $V - B - T$, E contains $|B - T|$ elements of $B - T$ and $V - E$ contains 0 elements of $B - T$, E contains 0 (resp. 1) element of $\{x\}$ and $V - E$ contains 1 (resp. 0) elements of $\{x\}$, E contains 1 (resp. $\mathbf{c}$) elements of $G``(\iota^p``V)$ and $V - E$ contains $\mathbf{c}$ (resp. 1) element of $G``(\iota^p``V)$, $V - E$ contains $|D - G``(\iota^p``V)|$ (resp. 0) elements of $D - G``(\iota^p``V)$ and E contains 0 (resp. $|D - G``(\iota^p``V)|$) elements of $D - G``(\iota^p``V)$, and E contains $|T - D| - 1$ (resp. 1) element of $T - D$, and $V - E$ contains 1 (resp. $|T - D| - 1$) element of $T - D$. This exhausting description should make it clear that $F``C$ is symmetric with support $\{B, D, T, G``(\iota^p``V), \{x\}\}$ and thus a set, so F is setlike. Then, for any setlike permutation f, $F(f(a)) =$

$B - \{x\} \cup \{G(\iota^p(f(a)))\}$ or $B \cup \{x\} - \{G(\iota^p(f(a)))\} =$

$B - \{x\} \cup \{G(f_p(\iota^p(a)))\}$ or $B \cup \{x\} - \{G(f_p(\iota^p(a)))\} =$

$B - \{x\} \cup \{f_{p+k}(G(\iota^p(a)))\}$ or $B \cup \{x\} - \{f_{p+k}(G(\iota^p(a)))\} =$

$f_{p+k+1}(B - \{x\} \cup \{G(\iota^p(a))\})$ or $f_{p+k+1}(B \cup \{x\} - \{G(\iota^p(a))\})$, recalling that B and $\{x\}$ are $(p+k+1)$-symmetric, which is in either case equal to $f_{p+k+1}(F(a))$.

Now the existence of an injection from V into A and of course the existence of the identity injection from A into V would suggest, if we were in a context where the Schröder-Bernstein theorem could be applied, the existence of a bijection from V onto A. Surprisingly, the usual construction of the map witnessing the Schröder-Bernstein theorem yields a setlike map here, though it is not at all obvious that this should be the case.

Let A be an infinite set. Let f be an injection from V into A. We partition the world into the sets $K_n = f^n``(V - A)$ and the sets $L_n = f^n``(A - f``V)$. The bijection F from the universe onto A is defined as f on the sets K_n and the identity on the sets L_n. It is clear that this is bijective and maps V onto A; it is altogether unclear why this should be setlike (our sets are closed under finite unions, but not under countable unions). But it is.

Let k be chosen so that $f(g(x)) = g_k(f(x))$ for all setlike bijections g. Observe that $f^n(g(x)) = g_{nk}(f^n(x))$ for all x and g. This means that for any fixed set B we must be able to choose n so that $f^n(g(x)) \in B \leftrightarrow f^n(x) \in B$ for all x and g (because for a large enough fixed n we will have $g_{nk}``B = B$), from which it follows that for all $x$ and $y$, $f_n(x) \in B$ iff $f_n(y) \in B$: all sufficiently iterated images under $f$ are either inside $B$ or outside $B$. This means that for the bijection $F$ given above and any fixed set $B$, $F``B$ is actually a set, because we only need to consider the ways in which B cuts finitely many of the K_n's and L_n's. For any fixed B, there is an N such that for all $n > N$, $K_n \cup L_n$ is either a subset of B or disjoint from B. $\bigcup_{n>N} K_n \cup L_n$ is a set (the complement of $\bigcup_{n \leq N} K_n \cup L_n$, which is a finite union of sets and so a set), and its image under F, $\bigcup_{n>N}(K_{n+1} \cup L_n)$, is similarly a set. $F``B$ is then either $\bigcup_{n \leq N} F``(B \cap (K_n \cup L_n))$ or $\bigcup_{n \leq N} F``(B \cap (K_n \cup L_n)) \cup \bigcup_{n>N}(K_{n+1} \cup L_n)$, depending on whether $\bigcup_{n>N} K_n \cup L_n$ is disjoint from B or included in B, and each of these is clearly a set.

This completes the proof that the extended structure (M^+, ϵ^+) is a model of our symmetric theory of sets and classes with $k = 1$, from which it follows by the results of the first part of the paper that (M, ϵ) is a model of *NFMSI*.

We note that it is possible to construct larger models of the symmetric theory with $k = 1$ and thus of *NFMSI* in a way which we very briefly indicate. Fix an inaccessible cardinal κ. Let the meaning of "metafinite" be "has cardinality less than κ". Construct a model precisely as above beginning with a set of cardinality 2^κ and a subset of its power set of size 2^κ with all elements either of size $< \kappa$ or of size 2^κ and with the splitting property for the largest sets. Build a type sequence in which X_{i+2} is the set of subsets of X_{i+1} which are (X_i, X_{i+1}) symmetric with a support S of cardinality $< \kappa$ taken from X_{i+1}. The construction and supporting proofs go in essentially the same way.

11 Questions

We have a few questions.

Two natural questions arise about the relations between the theories *NFSI* and *NFMSI* on the one hand and Grishin's NF_3 on the other. Does NF_3 entail *NFSI*? The latter theory is clearly very weak, and it would not surprise us at all of this implication held, but we do not see how to establish it. Clearly *NFMSI* entails NF_3. The consistency strength of *NFMSI* is no more than that of NF_3 + Infinity (the same as that of second-order arithmetic, a result of Pabion in [6]); this can be determined by examination of the construction of the previous section. Does *NFMSI* prove any theorem which is not a consequence of NF_3 (or perhaps NF_3+Infinity)?

A final question: is there a natural proof that choice does not hold in the set/class theories presented here? Of course if $k \geq 2$ we can use Specker's proof that choice is false in *NF*. But the role played by symmetry in the definition of these theories suggests that we ought to be able to disprove choice more directly.

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