Direct construction of a model of tangled type theory

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7/26/2017, 12:30 pm Boise time, negative indexed draft of the whole proof.

1 Version Notes

7/26/17: Various minor edits made.

General edits needed: the proof of the extension property needs to be articulated more carefully in terms of partial support maps rather than vague references to permutations and their derivatives acting as appropriate on suppot elements. What is said is correct, but the terminology of partial support maps allows it to be said precisely. Similar remarks apply in a couple of other places. 12:30 pm this point is addressed.

The reference to Crabbé's predicative NF paper is needed (addressed, 12:30 pm).

7/25/17: minor edits

7/23/17: This version has another bookkeeping item installed in elements of support sets which is required for the proof that the process of closing support sets up into strong supports actually works. The argument for this closure is given explicitly and I *hope* at last correctly. The basic idea is that a support element contains an additional annotation which will report if it is only needed for a sub-support of lower index and so does not need to have higher index closure conditions applied to it.

7/22/17: I think I have correctly reinstalled the case of litter elements as local cardinals in the strong support definition. The argument for existence of strong supports is still at a negative draft of large absolute value: details need to be filled in.

7/21/17: Refinement of the definition of strong support needed to ensure that strong supports exist. Some corrections still needed.

In this version, I have eliminated the case of litter elements as local cardinals from the definition of strong support. This could be reversed, but this would make the argument that there are strong supports more complicated. Eliminating this case makes the proof of the extension property slightly more complicated.

7/18/17: left out the case of a litter element being the local cardinal of a local cardinal element (in terms of the other approach, this is analogous to a regular atom being a parent of a near-litter).! Bookkeeping is a nightmare in all versions of this argument. Flurry of discoveries of silly typos in the preliminary 4:15 release.

3:05 pm, 7/18/17: added remark that extensionality obviously holds in M.

Contents

2 Introduction: definitions of type theories; equivalence of tangled type theory and NF

This paper presents a proof that Quine's New Foundations (the theory introduced in [9]) is consistent, by proving the consistency of a theory TTT_λ (tangled type theory with types indexed by ordinals less than a limit ordinal λ) which we showed in our paper [4] of 1995 to be equiconsistent with NF.

definition of simple typed theory of sets: TST, the simple typed theory of sets, is the first order theory with equality and membership with sorts indexed by the natural numbers. Atomic formulas $x = y$ are well-formed iff the sort of x is the same as the sort of y. Atomic formulas $x \in y$ are well-formed iff the sort of y is the successor of the sort of x. The axioms of TST are extensionality (objects of a positive type are equal iff they have the same elements) and comprehension (for any formula ϕ, $\{x^i : \phi\}^{i+1}$ exists).

definition of tangled type theory: We describe TTT_λ (theories of this kind were originally described in [4]). Fix a limit ordinal λ. TTT_λ is a first order theory with equality and membership with sorts indexed by ordinals $< \lambda$. Atomic formulas $x = y$ are well-formed iff the sorts of x and y are the same. Atomic formulas $x \in y$ are well-formed iff the sort of x is less then the sort of y. For any formula ϕ of the language of TST, and strictly increasing sequence s of ordinals $< \lambda$, obtain a formula ϕ^s by replacing type i variables with type $s(i)$ variables injectively in ϕ. The formula ϕ^s will be well-formed. The axioms of TTT_λ are of the form ϕ^s where ϕ is an axiom of TST. Note that this is *not* a cumulative type theory: each type is being interpreted as a "power set" (different higher types not necessarily as the same "power set") of each lower type, in such a way that every increasing sequence of types determines a model of TST.

tangled type theory and NF are equiconsistent: If NF is consistent, TTT_λ is consistent: disjoint copies of the model of NF with the membership relations between them induced by the membership relation of the model of NF in the obvious way yield a model of TTT_λ.

We claim that existence of a model of TTT_λ implies consistency of NF. This argument is as it were reverse engineered from Jensen's proof of the consistency of NFU (in [8]): this is our result in [4]. Let Σ be a finite set of formulas in the language of TST. Let n be larger than the largest type index appearing in Σ (so there are no more than n types appearing in Σ). Define a partition of $[\lambda]^n$ into $\leq 2^{|\Sigma|}$ compartments by placing a set A in a compartment determined by the truth values of the formulas ϕ^s in the given model of TTT_λ for $\phi \in \Sigma$ and s any strictly increasing sequence in λ whose first n elements are the elements of A. By Ramsey's theorem, this partition has an infinite homogeneous set H which includes the range of a strictly increasing sequence h. Interpret each sentence ϕ of the language of TST as ϕ^h: by assigning to ϕ the truth value of ϕ^h in the given model of TTT_λ, we get a complete consistent extension of TST in which the Ambiguity Scheme $\phi \leftrightarrow \phi^+$ of Specker holds for $\phi \in \Sigma$, whence it follows by compactness that the full Ambiguity Scheme is consistent with TST, whence it follows by the results of Specker in [14] that NF is consistent.

3 Construction of the model M

We work in ZFA with choice. The exact extent of the atoms will be described below.

ordinal and cardinal parameters of the construction: Fix a limit ordinal λ for the rest of the paper. Elements of λ may be called "type indices". Nonempty finite subsets of λ may be called "extended type indices". If A is an extended type index with at least two elements, A_1 is defined as $A \setminus \{\mathtt{min}(A)\}$ and A_{n+1} is defined as $(A_n)_1$ when this is defined. We make the technical restriction on extended type indices that any gaps in the natural order on the elements of the extended type index be of the form $(\beta + 2, \alpha + 2)$: the largest element below the gap and the smallest above the gap must both be double successor ordinals.

Fix an uncountable regular ordinal κ. Sets of cardinality $< \kappa$ will be termed "small" and other sets will be termed "large".

Fix a strong limit cardinal μ which is greater than λ, greater than κ, and has cofinality greater than or equal to κ.

preliminary description of the model M: We will build a structure M which is the disjoint union of sets M_α for $\alpha < \lambda$ whose description we commence. For each non-successor ordinal $\nu < \lambda$, M_ν is a collection of μ atoms. There is an atom $\emptyset_{\alpha+1}$ for each ordinal $\alpha < \lambda$: each $M_{\alpha+1}$ is a subset of $\mathcal{P}(M_\alpha) \setminus \{\emptyset\} \cup \{\emptyset_{\alpha+1}\}$. All atoms are either elements of M_ν's or $\emptyset_{\alpha+1}$'s. The set $M_{\alpha+2}$ will turn out to be type α of a model of tangled type theory; the relations $\in_{\beta,\alpha}$ of type $\beta < \alpha$ ($M_{\beta+2}$) in type α ($M_{\alpha+2}$) will be described; $\in_{\beta,\beta+1}$ is the appropriate restriction of the usual membership relation. Each set M_α is expected to be of size μ.

We will continue the convention that the letter ν is always used for non-successor ordinals less than λ. We define ν_α as the largest non-successor $\leq \alpha < \lambda$.

litters, near-litters and local cardinals: For each $\nu < \lambda$ non-successor, we choose a partition $\texttt{litters}_{\nu+1}$ of M_ν into sets of size κ. We refer to these sets as "litters". $\texttt{nearlitters}_{\nu+1}$ is defined as the collection of subsets of M_ν with small symmetric difference from a litter. For each ν, we stipulate that $\texttt{nearlitters}_{\nu+1} \subseteq M_{\nu+1}$. The elements of $\texttt{nearlitters}_{\nu+1}$ are referred to as "near-litters".

For each litter $L \in \texttt{litters}_{\nu+1}$, we define $[L]$, the local cardinal of L, as the set of all elements of $\texttt{nearlitters}_{\nu+1}$ with small symmetric difference from L. We define $K_{\nu+2}$ as the collection of all local cardinals of litters in $\texttt{litters}_{\nu+1}$. We stipulate that $K_{\nu+2} \subseteq M_{\nu+2}$. We partition $K_{\nu+2}$ into two sets, $K^a_{\nu+2}$ and $K^b_{\nu+2}$, each of size μ.[1]

For each $\nu < \lambda$, non-successor, we provide a partition $\texttt{litters}_{\nu+3}$ of $K^a_{\nu+2}$ into sets of size κ. We refer to these sets as well as litters. We define $\texttt{nearlitters}_{\nu+3}$ as the collection of all subsets of $K^a_{\nu+2}$ with small symmetric difference from a litter. We refer to these sets as "near-litters" as well. We stipulate that $\texttt{nearlitters}_{\nu+3} \subseteq M_{\nu+3}$. Further, we stipulate that

$$\{\bigcup N : N \in \texttt{nearlitters}_{\nu+3}\} \subseteq M_{\nu+2}.$$

Notice that for any $N \in \texttt{nearlitters}_{\nu+3}$, $\bigcup N$ uniquely determines N, since the elements of N belong to the pairwise disjoint collection $K_{\nu+2}$ of nonempty sets.

[1] In terms of the original construction, local cardinals are analogous to atoms, though they are sets here. Elements of $K^a_{\alpha+2}$'s correspond to regular atoms and elements of $K^b_{\alpha+2}$'s to irregular atoms.

For each $L \in \texttt{litters}_{\nu+3}$, define $[L]$, the local cardinal of L, as the collection of all $\bigcup N$ such that $N \in \texttt{nearlitters}_{\nu+3}$ has small symmetric difference from L. Notice the use of the set union operation to lower type by one. We define $K_{\nu+3}$ as the set of all $[L]$ for $L \in \texttt{litters}_{\nu+3}$. We stipulate that $K_{\nu+3} \subseteq M_{\nu+3}$. Note that $K_{\nu+3}$ is a pairwise disjoint collection of nonempty sets. We partition $K_{\nu+3}$ into two sets, $K^a_{\nu+3}$ and $K^b_{\nu+3}$, each of size μ.

Let $\alpha < \lambda$. Assume that $K_{\alpha+3}$ (and so its moiety $K^a_{\alpha+3}$) have been defined and are pairwise disjoint collections of elements of $M_{\alpha+3}$.

We provide a partition $\texttt{litters}_{\alpha+4}$ of $K^a_{\alpha+3}$ into sets of size κ. We refer to these sets as well as litters. We define $\texttt{nearlitters}_{\alpha+4}$ as the collection of all subsets of $K^a_{\alpha+3}$ with small symmetric difference from a litter. We refer to these sets as "near-litters" as well. We stipulate that $\texttt{nearlitters}_{\alpha+4} \subseteq M_{\alpha+4}$. Further, we stipulate that

$$\{\bigcup N : N \in \texttt{nearlitters}_{\alpha+4}\} \subseteq M_{\alpha+3}.$$

Notice that for any $N \in \texttt{nearlitters}_{\alpha+4}$, $\bigcup N$ uniquely determines N, since the elements of N belong to the pairwise disjoint collection $K_{\alpha+3}$ of nonempty sets.

For each $L \in \texttt{litters}_{\alpha+4}$, define $[L]$, the local cardinal of L, as the collection of all $\bigcup N$ such that $N \in \texttt{nearlitters}_{\alpha+4}$ has small symmetric difference from L. Notice the use of the set union operation to lower type by one. We define $K_{\alpha+4}$ as the set of all $[L]$ for $L \in \texttt{litters}_{\alpha+4}$. We stipulate that $K_{\alpha+4} \subseteq M_{\alpha+4}$. Note that $K_{\alpha+4}$ is a pairwise disjoint collection of nonempty sets. We partition $K_{\alpha+4}$ into two sets, $K^a_{\alpha+4}$ and $K^b_{\alpha+4}$, each of size μ.

terminology and notation related to litters, near-litters and local cardinals: We introduce some useful terminology. Elements of sets $K^a_{\alpha+2}$ are referred to as litter elements (because they are the local cardinals which belong to litters). Elements of sets $K^b_{\alpha+2}$ might be referred to as "free junk". Elements of local cardinals are more briefly referred to as local cardinal elements; these are either near-litters whose elements are atoms or set unions of near-litters whose elements are local cardinals.

We define some useful operations on elements of local cardinals. If N is an element of a local cardinal, it is either equal to a near-litter (if its elements are atoms) or the set union of a near-litter (if its elements are local cardinals); in either case, we denote this near-litter by N^+. The litter with small symmetric difference from N^+ we denote by N° (if N is a general near-litter, N° denotes the litter with small symmetric difference from N). The local cardinal $[N^\circ]$ we abbreviate as $[N]$. For any local cardinal element N, we refer to the elements of $N^+\Delta N^\circ$ as the anomalies of N (and also as the anomalies of N^+).

collections of free junk objects reserved for later purposes: We further designate some subsets of the sets $K^b_{\gamma+2}$ for future use: for each α, β with $\beta+1 < \alpha$, we provide a set of size μ included in $K^b_{\alpha+2}$ which we call $\texttt{rng}(\sigma_{\beta+2,\alpha+2})$ and we provide a subset of $K^b_{\beta+3}$ of size μ which we call $\texttt{rng}(\tau_{\alpha+1,\beta+3})$. All of these sets are disjoint from one another.

Sets $M_{\nu+1}$ specified: Let $\nu < \lambda$ be a non-successor. $M_{\nu+1}$ is the collection of sets with small symmetric difference from set unions of small or co-small subsets of $\texttt{litters}_\nu$. Note that $M_{\nu+1}$ is of cardinality μ.

We discuss how $M_{\alpha+2}$ is to be constructed under the inductive hypotheses that each M_β for $\beta < \alpha+2$ has been constructed successfully and is of size μ, and that if $\beta = \gamma+2 < \alpha+2$ is a double successor, M_β was constructed in the same way described here. Note that certain elements of $M_{\alpha+2}$ (near-litters, local cardinals and local cardinal elements) have already been constructed.

Embedding a "power set" of $M_{\beta+2}$ into $\bigcup$ "$\mathcal{P}(\texttt{rng}(\sigma_{\beta+2,\alpha+2})$: We note that $\bigcup$ "X denotes $\{\bigcup x : x \in X\}$.

Our intention is to create a sense in which $M_{\alpha+2}$ is interpretable as a "power set" of each $M_{\beta+2}$ where $\beta+1 < \alpha$ (where $\beta+1 = \alpha$ ordinary membership relations in M handle this). The initial idea is to provide a bijection $\sigma_{\beta+2,\alpha+2}$ from $M_{\beta+2}$ to $\texttt{rng}(\sigma_{\beta+2,\alpha+2}) \subseteq K^b_{\alpha+2} \subseteq M_{\alpha+2}$. We then associate each element A of $M_{\beta+3}$ (or more generally each $A \subseteq M_{\beta+2}$) with $\bigcup(\sigma_{\beta+2,\alpha+2}``A) \subseteq M_{\alpha+1}$ (that this image belongs to $M_{\alpha+2}$ will be demonstrable for $A \in M_{\beta+3}$ satisfying a technical condition revealed below; some sets in $M_{\beta+3}$ will not be replicated). The representation of our first approximation to a "power set of $M_{\beta+2}$" in $M_{\alpha+2}$ will include some other set unions of subsets of the range of $\sigma_{\beta+2,\alpha+2}$ (all the other such unions which turn out to belong to $M_{\alpha+2}$): this "power set" will have some elements not corresponding to elements of the original "power set" $M_{\beta+3}$ of $M_{\beta+2}$. These first approximations to "power sets" are of course not genuine power sets: they do not contain all subsets of $M_{\beta+2}$, and will turn out to be of cardinality μ, the same size as $M_{\beta+2}$.

Representing a "power set" of $M_{\beta+2}$ by all of $M_{\alpha+2}$: We want our "power set of $M_{\beta+2}$" to coincide with $M_{\alpha+2}$, not be confined to the collection $\bigcup$ "$\mathcal{P}(\mathtt{rng}(\sigma_{\beta+2,\alpha+2}) \cap M_{\alpha+2}$ of set unions of subsets of $\mathtt{rng}(\sigma_{\beta+2,\alpha+2})$ which happen to belong to $M_{\alpha+2}$. To arrange this, we ensure that the collection of set unions of subsets of the range of $\sigma_{\beta+2,\alpha+2}$ belonging to $M_{\alpha+2}$ is as large as the whole of $M_{\alpha+2}$ in a suitable sense. We do this by providing a bijection $\tau_{\alpha+1,\beta+3}$ from the whole of $M_{\alpha+1}$ to $\mathtt{rng}(\tau_{\alpha+1,\beta+3}) \subseteq K^b_{\beta+3} \subseteq M_{\beta+3}$. This enables us to map an arbitrary $A \in M_{\alpha+2}$ to a union of elements of the range of $\sigma_{\beta+2,\alpha+2}$: $f_0^{\alpha,\beta}(A) = \bigcup \sigma_{\beta+2,\alpha+2}$"$\bigcup(\tau_{\alpha+1,\beta+3}$"$A) \subseteq M_{\alpha+1}$ (membership of this set in $M_{\alpha+2}$ will be demonstrable). After we complete the definition of $M_{\alpha+2}$ we will obtain a bijection $f^{\alpha,\beta}$ from $M_{\alpha+2}$ onto the collection of set unions of subsets of the range of $\sigma_{\beta+2,\alpha+2}$ which happen to belong to $M_{\alpha+2}$: send each element of $M_{\alpha+2}$ which is not such a union to its image under $f_0^{\alpha,\beta}$, and send each iterated image of such an element under $f_0^{\alpha,\beta}$ to its image under $f_0^{\alpha,\beta}$, and fix all other such unions: this is the standard Schröder-Bernstein trick.

The membership relation of the model of tangled type theory defined: The relation $x \in_{\beta,\alpha} y$ of the eventual model of tangled type theory is defined, for $x \in M_{\beta+2}$ and $y \in M_{\alpha+2}$, where $\beta + 1 < \alpha$, as $\sigma_{\beta+2,\alpha+2}(x) \subseteq f^{\alpha,\beta}(y)$. There is considerable work to be done to complete our definition of this relation and formulate and verify our exact claims about it.

Note that the ranges of the σ and τ maps were set aside earlier in the construction and they are known to be disjoint from one another.

We now commence discussion of permutations and supports.

Definition of allowable permutations: We now define a family of permutations of interest, the $\alpha + 2$-allowable permutations. An $\alpha + 2$-allowable permutation is a permutation ρ of the set of atoms M_{ν_α}; the action of a permutation on sets in $M_{\nu_\alpha+i}$'s is defined by the rules $\rho(A) = \rho$"A and $\rho(\emptyset_{\gamma+1}) = \emptyset_{\gamma+1}$. Further, an allowable permutation sends litter elements and local cardinal elements belonging to any M_β with $\nu_\alpha < \beta \leq \alpha + 2$ to litter elements and local cardinal elements, respectively (necessarily in the same M_β), and so does its inverse.

Note that specification of the action of an allowable permutation on any $M_{\nu_\alpha+i}$ induces its definition on M_ν: consider its action on iterated singletons of atoms.

There are further stipulations relating $\alpha+2$-allowable permutations to the σ and τ maps defined above. For any $\alpha+2$-allowable permutation ρ, ρ commutes with all the maps $f_0^{\beta,\gamma}$ where $\gamma+1 < \beta \leq \alpha$ and $\nu_\alpha \leq \beta$. [we are working under the inductive hypothesis that we have carried out the same process we are describing at all earlier stages], and further, where $\gamma + 1 < \beta \leq \alpha$ and $\nu_\alpha \leq \beta$, for each $\sigma_{\gamma+2,\beta+2}$ we have that

$$\sigma^{-1}_{\gamma+2,\beta+2} \circ \rho \circ \sigma_{\gamma+2,\beta+2}$$

is always defined and equivalent on its domain to the action of a $\gamma+2$-allowable permutation, which we will write $\rho_{\gamma+2,\beta+2}$ ($\gamma + 2$-allowable permutations are supposed already to have been defined). We extend this notation by allowing $\rho_{\beta,\beta+1}$ to denote ρ.

It is worth noting that there is no constraint on the action of $\rho_{\beta+2,\alpha+2}$, a $\beta+2$-allowable permutation, on local cardinal elements in $M_{\beta+2}$ (the latter being elements of local cardinals in $M_{\beta+3}$), other than that they be mapped to local cardinal elements; the local cardinals to which they belong are in the "gap" between $\beta + 2$ and $\alpha + 2$ and so are not considered. The permutation $\rho_{\beta+2,\alpha+2}$ does induce a permutation of local cardinals in $M_{\beta+3}$, but an entirely arbitrary one: it is evident that a permutation sending local cardinal elements to local cardinal elements, whose inverse has the same property, maps the elements of a fixed local cardinal onto the elements of another (or the same) fixed local cardinal. Similarly, an $\alpha + 2$-allowable permutation induces a permutation of the local cardinals in $M_{\alpha+3}$, since it sends local cardinal elements in $M_{\alpha+2}$ to local cardinal elements, but this permutation is entirely arbitrary.

This completes the definition of $\alpha + 2$-allowable permutations.

derivatives of allowable permutations: For each $\alpha+2$-allowable permutation ρ and suitable extended type indices A (ones with $\texttt{max}(A) =$

$\alpha + 2$) we define permutations ρ_A which we call "derivatives" of ρ. ρ_A is the $\mathtt{min}(A)$-allowable permutation defined as $(\rho_{A_1})_{\mathtt{min}(A),\mathtt{min}(A_1)}$ when A has at least two members and the two smallest members are not successive. When $\mathtt{min}(A)$ and $\mathtt{min}(A_1)$ are successive, $\rho_A = \rho_{A_1}$ (this clause also follows from the meaning assigned to $\rho_{\beta,\beta+1}$ above); $\rho_{\{\alpha\}} = \rho$. It is important to note that there is no reason to believe that derivatives of ρ with distinct indices acting on the same M_γ's agree with one another, except when one index downward extends the other with an interval of additional elements (no gaps in the additional elements).

We can now develop the basic property of τ maps.

the relationship between allowable permutations and the τ maps: If ρ is $\alpha + 2$-allowable and $\gamma + 2 < \beta \leq \alpha$, we have $\rho(f_0^{\beta,\gamma}(\{x\})) = f_0^{\beta,\gamma}(\rho(\{x\}))$.

Recall that $f_0^{\beta,\gamma}(\{x\}) = \bigcup \sigma_{\gamma+2,\beta+2}``\bigcup(\tau_{\beta+1,\gamma+3}``\{x\})$.

Thus

$$\rho(f_0^{\beta,\gamma}(\{x\}))$$
$$= \rho(\bigcup \sigma_{\gamma+2,\beta+2}``\bigcup(\tau_{\beta+1,\gamma+3}``\{x\}))$$
$$= \bigcup \sigma_{\gamma+2,\beta+2}``(\rho_{\gamma+2,\beta+2}(\bigcup(\tau_{\beta+1,\gamma+3}``\{x\}))$$
$$= \bigcup \sigma_{\gamma+2,\beta+2}``(\rho_{\gamma+2,\beta+2}(\tau_{\beta+1,\gamma+3}(x)),$$

and

$$f_0^{\beta,\gamma}(\rho(\{x\}))$$
$$= \bigcup \sigma_{\gamma+2,\beta+2}``\bigcup(\tau_{\beta+1,\gamma+3}``\{\rho(x)\})$$
$$= \bigcup \sigma_{\gamma+2,\beta+2}``(\tau_{\beta+1,\gamma+3}(\rho(x)),$$

so we have the identity $\tau_{\beta+1,\gamma+3}(\rho(x)) = \rho_{\gamma+2,\beta+2}(\tau_{\beta+1,\gamma+3}(x))$.

At this point it is worth observing that the action of general $\gamma + 2$-allowable permutations not of the exact form $\rho_{\gamma+2,\beta+2}$ on elements of local cardinals $\tau_{\beta+1,\gamma+3}(x)$ is quite arbitrary: they are of course sent to local cardinal elements but there is no other control.

Note also that the property of the τ maps presented is sufficient to prove the commuting maps condition from which it has been derived here.

ν- and $\nu+1$-allowable permutations: We stipulate that ν-allowable permutations are arbitrary permutations of M_ν, and that $(\nu+1)$-allowable permutations are permutations of M_ν whose actions fix $K_{\nu+2}$. We note that ν-supports are trivial (singletons of atoms) and $\nu+1$ supports are small sets of atoms in M_ν and near-litters in $M_{\nu+1}$, obtainable from the representation of an element of $M_{\nu+1}$ as having small symmetric difference (whose elements go in the support) from a small or co-small union of litters (the litters in the small set whose union is taken or the complement of the co-small set whose union is taken go in the support).

definition of support, action of allowable permutations on supports: Next, we present a definition of support. An $\alpha+2$-support set is a small set of triples (x, A, γ) where $x \in M_{\mathtt{min}(A)}$, $\mathtt{max}(A) = \alpha+2$, and x is either an atom, an element of $K^a_{\mathtt{min}(A)}$ (a litter element) or an element of $\bigcup K_{\mathtt{min}(A)+1}$ (a local cardinal element); we also require that elements of local cardinals (near-litters or unions of near-litters) M and N which appear in pairs in S with the same extended type index are disjoint if distinct, and that there cannot be two elements of an $\alpha+2$-support set which differ only in their third components. γ is an ordinal element of A (so $\mathtt{min}(A) \leq \gamma \leq \alpha+2$); its function will be seen in the definition of strong support below.

For any $\alpha+2$-allowable permutation ρ and triple (x, A, γ) satisfying the conditions above, we define $[\rho](x, A, \gamma)$ as $(\rho_A(x), A, \gamma)$. We define $\rho[S]$, where S is a support set as defined above, as $[\rho]``S$.

We say that S is a $\alpha+2$-indexed support for an object X iff S is an $\alpha+2$-support set and each $\alpha+2$-allowable permutation ρ satisfying $[\rho](x, A, \gamma) = (x, A, \gamma)$ for each $(x, A, \gamma) \in S$ such that $\gamma = \alpha+2$ also satisfies $\rho(X) = X$.

Definition of strong supports and support orders: A strong $\alpha+2$-support for an object X is an $\alpha+2$ support S for X which supports a well-ordering $<_S$ with the following closure properties:

1. Any triple (x, A, γ) in S where x is an atom or local cardinal belonging to a litter is preceded in $<_S$ by a triple (N, A_1, δ) or (N, A, δ) as appropriate, where N is a local cardinal element and includes x as an element (if x is an atom) or a subset (if x is a local cardinal) and $\delta \geq \gamma$.

2. Any triple (N, A, γ) where N is an element of a local cardinal and $[N]$ is a litter element, A has at least two elements, $\gamma \geq \mathtt{min}(A_1)$, and the two smallest elements of A are successive, is preceded in $<_S$ by a triple $([N], A_1, \delta)$ with $\delta \geq \gamma$.
3. Any triple (N, A, χ) where N is an element of a local cardinal and $[N] = \sigma_{\gamma+2,\beta+2}(X)$ with $\beta + 2 \leq \alpha + 2$ and with $\beta + 1, \beta + 2$ as the two smallest elements of A and with $\chi \geq \beta + 2$ will be preceded in $<_S$ by a collection of items produced from a $\gamma + 2$-support for X by replacing each element (s, B, χ') of the support with a triple $(s, B \cup A_1, \delta))$ with $\gamma + 2 \leq \delta$.
4. A triple (N, A, χ) where N is an element of a local cardinal, $[N] = \tau_{\beta+1,\gamma+3}(X)$, where $\beta + 1 < \alpha + 2$, and which has the two smallest elements of A equal to $\gamma+2$ and $\beta+2$ will be preceded by elements obtained from a $\beta + 1$-support for X by replacing each element (s, B, χ') with a triple $(s, B \cup A_1, \delta)$ with $\beta + 1 \leq \delta$.

Note that if S is a strong $\alpha + 2$-support for X then $\rho[S]$ is a strong $\alpha + 2$-support for $\rho(X)$ with order $<_{\rho[S]}$ constructed in the obvious way by application of $[\rho]$; this is straightforward to verify from the relationships between ρ and the σ and τ maps.

The discussion of permutations and supports here should be reminiscent of Fraenkel-Mostowski methods for independence results for choice-related statements (see [7] for a discussion), though with mysterious additional bookkeeping, and indeed other versions of this proof have been presented explicitly in terms of FM methods, but in this approach the use of permutations is sufficiently exotic that a self-contained proof that the method works is given in a later section.

purely motivational remark about extended type indices: The idea here is that the tangled type theory is to be thought of as "unfolded", each M_α being supposed to be unfolded into many copies M_A with A an extended type index with $\mathtt{min}(A) = \alpha$, with the "power set" of M_A being understood to be M_{A_1}, with the "membership relation" of M_A in M_{A_1} being determined by the relation $\in_{\mathtt{min}(A)-2,\mathtt{min}(A_1)-2}$ (ordinary membership where these indices are successive). The idea is to make the impossible power set structure of the model of tangled type theory, in which each M_α is in some sense the power set of each M_β

with $\beta < \alpha$, invisible. Support elements are tagged to indicate which "branch" of this ramified type structure they are considered to be on. The derivative permutations are used to permute in different ways different branches of the ramified type structure which are at bottom the same, so that their identity with one another is obscured.

The third item in a support triple indicates what kind of support this item is supposed to belong to, and can be used to prevent the need for application of closure conditions appropriate to a higher indexed support to that item.

Definition of $M_{\alpha+2}$ and so of M completed (mod a proof obligation): We define $M_{\alpha+2}$ as consisting of $\emptyset_{\alpha+2}$ and all subsets of $M_{\alpha+1}$ which have $\alpha + 2$-support, as long as this collection is of size μ [this is what we mean by the construction being successful at earlier stages in the statement of the inductive hypotheses above]: otherwise we stipulate that $M_{\alpha+2}$ contains only $\emptyset_{\alpha+2}$ and the local cardinals, near-litters, and local cardinal elements already stipulated to belong to this set, and in this case we say that the construction fails and we stipulate that it fails similarly at all subsequent stages.

This completes the definition of the structure M.

Final observation: To complete the proof that the construction succeeds (that each $M_{\alpha+2}$ actually contains all subsets of $M_{\alpha+1}$ with $\alpha + 2$-support) we need to verify that the collection of subsets of $M_{\alpha+2}$ with $\alpha + 1$-support has no more than μ elements (it clearly has at least μ elements: consider iterated singletons of atoms).

4 Verification of required properties of M

Supports in which all (set unions of) near-litters are (set unions of) litters: In constructing a support set, we can ensure that elements of local cardinals which appear are always litters or unions of litters, by replacing any near-litter N or union $\bigcup N$ of a near-litter with L or $\bigcup L$, where L is the litter with small symmetric difference from N, along with the elements of $N \Delta L$.

Existence of strong supports: We claim that any $\alpha + 2$-support set can be extended to a strong support set with an accompanying order.

For each element x of any $M_{\beta+2}$ for $\beta < \alpha$, we designate a $\beta+2$-support of x in which all elements have third component $\beta + 2$: such a support exists by our hypothesis that the construction succeeded at all earlier stages.

We say that a support set element (x, A, γ) *requires* a support set element (y, B, δ) essentially if the closure conditions for a strong support, along with the use of the designated support for x and the condition that all near-litters or unions of near-litters in the support be litters or unions of litters, require that a strong support set containing (x, A, γ) contain some (y, B, δ') with $\delta' \geq \delta$:

1. Any triple (x, A, γ) in S where x is an atom or local cardinal belonging to a litter requires (L, A_1, γ) or $(\bigcup L, A, \gamma)$ as appropriate, where L is the litter containing x.
2. Any triple (N, A, γ) where N is an element of a local cardinal and $[N]$ is a litter element, A has at least two elements, $\gamma \geq \mathtt{min}(A_1)$, and the two smallest elements of A are successive, requires $([N], A_1, \gamma)$.
3. Any triple (N, A, χ) where N is an element of a local cardinal and $[N] = \sigma_{\gamma+2,\beta+2}(X)$ with $\beta + 2 \leq \alpha + 2$ and with $\beta + 1, \beta + 2 \leq \chi$ as the two smallest elements of A requires $(s, B \cup A_1, \gamma + 2)$ for each $(s, B, \gamma + 2)$ in the designated $\gamma + 2$-support for $[N]$.
4. A triple (N, A, χ) where N is an element of a local cardinal, $[N] = \tau_{\beta+1,\gamma+3}(X)$, where $\beta + 1 < \alpha + 2$, and which has the two smallest elements of A equal to $\gamma + 2$ and $\beta + 2$ requires $(s, B \cup A_1, \beta + 1)$ for each $(s, B, \beta + 1)$ in the designated $\beta + 1$-support of $[N]$.

We claim that if we start with any $\alpha + 2$-support set and go through ω steps, at each step adding all elements required by the elements present at the previous step (where two distinct triples differing only in their third components are required, choosing the one with the larger third component), we will obtain an $\alpha + 2$-strong support set.

The basic reason for this is that there can be no infinite sequence $\{(x_i, A_i, \gamma_i)\}_{i \in \mathbb{N}}$ in which (x_i, A_i, γ_i) requires $(x_{i+1}, A_{i+1}, \gamma_{i+1})$ for each i. We show this for sequences starting with a given (x, A, γ) under the inductive hypothesis that this holds for any triple (y, B, δ) in which either $|B| < |A|$ or $\mathtt{max}(B) < \mathtt{max}(A)$.

We call a finite or infinite sequence in which (x_i, A_i, γ_i) requires $(x_{i+1}, A_{i+1}, \gamma_{i+1})$ for each i for which i and $i+1$ are in its (possibly finite) domain a bad sequence.

To begin with, our claim is true if $|A| = 1$, since in this case (x, A, γ) can only require (L, A_1, γ) or $(\bigcup L, A, \gamma)$ as appropriate, and neither of these will require any further triple if A has only one element.

It holds by inductive hypothesis if x is a local cardinal element and $[x]$ is a litter element. It holds trivially if x is a local cardinal element and (x, A, γ) does not require any further triple.

It holds for all triples if it holds for triples where x is a local cardinal element, since a triple where x is not a local cardinal element requires only a single triple whose first component is a local cardinal element.

So all we need to show is that there is no infinite bad sequence starting with any triple (N, A, χ) where N is an element of a local cardinal and $[N] = \sigma_{\gamma+2,\beta+2}(X)$ with $\beta + 2 \leq \alpha + 2$ and with $\beta + 1, \beta + 2 \leq \chi$ as the two smallest elements of A , or any triple (N, A, χ) where N is an element of a local cardinal, $[N] = \tau_{\beta+1,\gamma+3}(X)$, where $\beta + 1 < \alpha + 2$, and which has the two smallest elements of A equal to $\gamma + 2$ and $\beta + 2$.

The second element of any such bad sequence will be of the form $(s, A_1 \cup B, \delta)$ where s belongs to the designated δ-support for $[N]$. A bad sequence starting with $(s, A_1 \cup B, \delta)$ will have an initial segment which is obtained from the terms of a bad sequence starting at (s, B, δ) by replacing each second component with its union with A_1. This sequence will be finite by inductive hypothesis, so will have a last term (t, C, δ') corresponding to a term $(t, A_1 \cup C, \delta')$ of the original bad sequence. t cannot be an atom or litter element. The term $(t, A_1 \cup C, \delta')$ can only require a further term if C has one element. If $[t]$ were a litter element, we have argued above that our bad sequence cannot be infinite by inductive hypothesis. There are now two cases. If the two smallest elements of $A_1 \cup C$ are successive, and if $\delta' \geq \texttt{min}(A_1)$, the next term is of the form $(u, A_1 \cup D, \texttt{max}(D))$ where the minimum of A_1 and $\texttt{max}(D)$ are not successive. Further terms of the bad sequence will eventually arrive at a $(v, A_1 \cup E, \delta'')$, where there is no mechanism for δ'' to be anything but the sole member of E. In this case, if the two smallest elements of $A_1 \cup E$ are successive, the bad sequence terminates. So the only case we have to consider is that in which we have arrived at a term $(t, A_1 \cup C, \delta')$ and the two smallest elements of $A_1 \cup C$ are not successive. In this case, any triples required by $(t, A_1 \cup C, \delta')$ are of the form $(u, A_1 \cup \{\mathtt{min}(A_1) - 1\} \cup E, \mathtt{min}(A_1) - 1)$, and further extension will lead to something of the form $(v, A_1 \cup \{\mathtt{min}(A_1) - 1\} \cup C, \delta''')$, where C has a single element. Iteration of this process through an infinite bad sequence will adjoin successive smaller elements to the index indefinitely, which is impossible.

This result is enough both to show that ω closures under requirement are enough to ensure the desired closure conditions and that there is no obstruction to all items required by a given item preceding that item in an order on the support.

Definition and statement of the extension property for partial support maps: We now articulate and prove a result that the $\alpha + 2$-allowable permutations act fairly freely.

Define a partial support map $[\rho]^0$ as a map acting on pairs (x, A) where $x \in M_{\mathtt{min}(A)}$, A is an extended type index with maximum $\alpha + 2$, and x is either an atom or a local cardinal, satisfying the following conditions (in which we define $\rho^0_A(x) = y$ iff $[\rho]^0(x, A) = (y, A)$):

1. The second projection of $[\rho]^0(x, A)$ is always A.
2. Each ρ^0_A has domain equal to its range.
3. The domain of ρ^0_A does not include any element of the range of a σ map on which the derivative ρ_A has obligations: if $\gamma + 2 < \beta + 2$ are the two smallest elements of A, no element of $\mathtt{rng}(\sigma_{\gamma+2,\beta+2})$ is in the domain of ρ_A.
4. The intersection between the domain of ρ^0_A and any litter is small (empty being a special case of small). $\rho^0_A(x)$ is a litter element iff x is a litter element, for appropriate x.

We define a derivative partial support map $[\rho_A]^0$ as sending (x, B) to $(\rho^0_{A_1 \cup B}(x), B)$ where all elements of A_1 strictly dominate all elements of B, $\mathtt{max}(B) = \mathtt{min}(A)$, and $\rho^0_{A_1 \cup B}(x)$ is defined. $[\rho_{\beta+2,\alpha+2}]^0$ abbreviates $[\rho_{\{\beta+2,\alpha+2\}}]^0$.

The statement of the extension property for partial support maps is that for any partial support map $[\rho]^0$ there is an $\alpha + 2$-allowable permutation ρ such that ρ_A extends ρ^0_A (defined as above) for each A, which further has the property that the only "exceptions" of any ρ_A are elements of the domain of ρ^0_A, an exception of a γ-allowable permutation ρ being an element x of a litter L such that $x \in L^\circ$ and $\rho(x) \in \rho(L)^\circ$ have different truth values (N° being the litter with small symmetric difference from the near-litter N), and further $x \in M_\delta$ with $\delta \leq \gamma$.

proof of the extension property for partial support maps: Given the family of maps ρ^0_A determind by an $\alpha+2$-partial support map $[\rho]^0$, we compute the maps ρ_A extending them, the family of derivatives of a fixed $\alpha+2$-allowable permutation ρ, by a recursion along $\alpha+2$-strong supports in which any elements of local cardinals are litters or unions of litters (with due attention to making sure that the value at any given object is independent of the support of that object used). We assume that the result has already been shown to hold for all $\beta+2 < \alpha+2$. (The result holds trivially for ν- and $\nu+1$-allowable permutations, which have no derivatives). We further provide for every pair of litters L, M and appropriate extended type index A a map $\rho^{L,M}_A$ sending $L \setminus \texttt{dom}(\rho^0_A)$ to $L \setminus \texttt{dom}(\rho^0_A)$ bijectively. We will contruct the extension ρ_A so as to extend all the maps $\rho^{L,\rho(L)^\circ}_A$.

We describe a general procedure for computing $\rho(x)$. We presume that we have already computed $\rho_A(y)$ for each (y, A, γ) in an $\alpha+2$-strong support S of x (and we will also discuss how to compute $\rho_A(y)$ on the assumption that we have computed each $\rho_B(z)$ where $(z, B, \gamma) <_S (y, A, \delta)$). We do assume that all near-litters or unions of near-litters which are first components of triples in S are actually litters or unions of litters.

If $x \in M_\beta$ is a litter element ($\nu_\alpha \leq \beta$), we have already computed $\rho(L)$ for a litter L which contains x; we compute $\rho(x)$ as either $\rho^0_{[\beta,\alpha+2]}(x)$[2] or $\rho^{L,\rho(L)^\circ}_{[\beta,\alpha+2]}$ (where N° is the litter L such that $[L] = [N]$; note a fact which we exploit below, that if x is the union of a litter that litter is x°, and we know $\rho(x^\circ)$ iff we know $\rho(x)$). If x is a litter or union of a litter, and $[x]$ is not a litter element or an image under a relevant σ map, then we map $[x]$ to $\rho^0_{[\beta,\alpha+2]}([x])$ if this exists and otherwise to $[x]$,

[2]The special case $\beta = \alpha+2$ is possible here: the point we are making is that the extended type index has no gaps in it: if it did, our obligation would not be to compute ρ but a nontrivial derivative.

and then compute $\rho(x^\circ)$ by applying to each $y \in x^\circ$ the appropriate one of $\rho^0_{[\beta,\alpha+2]}$ or $\rho^{x^\circ,\rho(x)^\circ}_{[\beta,\alpha+2]}$, recalling that $\rho(x)^\circ$ is already known without calculation of $\rho(x)$ as the litter L such that $[L] = \rho([x])$. If $[x]$ is a litter element in M_β for $\beta < \alpha+2$; we have already computed $\rho([x])$ because it appears earlier in the support, and we compute $\rho(x)$ given $\rho([x])$ just as above. If $x \in M_{\beta+1}$ and $[x]$ is $\sigma_{\beta+2,\alpha+2}(X)$, then we compute $\rho([x])$ as $\sigma_{\beta+2,\alpha+2}(\rho_{\beta+2,\alpha+2}(X))$, where we can compute this last by inductive hypothesis, since $\rho_{\beta+2,\alpha+2}$ is a $\beta+2$-allowable permutation for which we have suitable data as a subset of the data we started with for ρ: we have already computed values for a partial support map (a restriction of $[\rho_{\beta+2,\alpha+2}]^0$) at each element of an indexed $\beta+2$-support of X (that is, we have computed values of $\rho_{\beta+2,\alpha+2}$ or appropriate derivatives at the first element of each triple in such a support); by inductive hypothesis, we can construct an allowable permutation ρ' by applying the extension property to the given partial support map restricting $[\rho_{\beta+2,\alpha+2}]^0$, and report $\sigma_{\beta+2,\alpha+2}(\rho'(X))$ as the value of $\rho([x])$ (any function ρ' with the correct values and values of its derivatives at elements of the support will give the same value at X). We then compute $\rho(x^\circ)$, given $\rho([x])$, by applying to each $y \in x^\circ$ the appropriate one of $\rho^0_{[\beta+1,\alpha+2]}$ or $\rho^{x^\circ,\rho(x)^\circ}_{[\beta+1,\alpha+2]}$. It is important to note here that if we are computing $\rho = \rho_{[\beta+1,\alpha+2]}$ at $x \in M_{\beta+1}$ we have $\beta+1, \beta+2$ the two smallest elements of the extended type index associated with x (and we assume that the third component of the support element is $\geq \beta+2$), so we do indeed have the previous data we claimed to have above, because we are using a strong support; if the extended type index had a gap above $\beta+1$, then we would be computing a nontrivial derivative of ρ whose value at x would be essentially arbitrary.

We need to consider computation of values $\rho_A(y)$ in the support of x on the assumption that we have computed all values of $\rho_B(z)$ for $(z, B, \gamma) <_S (y, A, \delta)$. In all cases but one this is straightforward; if $A = \{\alpha+2\}$ (or any $[\beta, \alpha+2]$) we are in the case above. Otherwise, in most cases, we are simply computing a permutation ρ_A which is of a lower index $\mathtt{min}(A) < \alpha+2$, and for which we have sufficient data to compute the value by inductive hypothesis. The one weird case is that of calculation of values $\rho_{\beta+2,\alpha+2}(x)$ for $x \in M_{\beta+2}$ and $[x] = \tau_{\alpha+1,\beta+3}(X)$ for some $X \in M_{\alpha+1}$. This has to do with the equation $\tau_{\alpha+1,\beta+3}(\rho(x)) = \rho_{\beta+2,\alpha+2}(\tau_{\alpha+1,\beta+3}(x))$. Thus if $[x] = \tau_{\alpha+1,\beta+3}(X)$ our calculation of

$\rho_{\beta+2,\alpha+2}([x])$ as $\tau_{\alpha+1,\beta+3}(\rho(X))$ appeals to another value of ρ (at $x \in M_{\alpha+1}$). Then the key is that the structure of strong supports ensures that we have already computed values of a partial support function (a restriction of $[\rho_{\{\alpha+1,\alpha+2\}}]^0$) at the elements of a $\alpha+1$-support of x, sufficient to compute the value of $\rho(x)$ as in the previous case (so what we do is compute a $\alpha+1$-allowable permutation which approximates ρ well enough, noting $\alpha+1 < \alpha+2$, and use its value at x). The method of computation of $\rho_{\beta+2,\alpha+2}(x)$ given $\rho_{\beta+2,\alpha+2}([x])$ has already been described.

The issue of whether calculation of $\rho(x)$ depends on the choice of strong support for x is settled quickly: the only case in which it might appear that the choice of strong support might make a difference is in the calculation of approximations to ρ or its derivatives using values previously computed at supports: but these are allowable permutations of lower index: if values do not depend on support used for all lower indices, they do not depend on support used for index $\alpha+2$.

The claim about exceptions is evident from the role of the maps $\rho_A^{L,M}$ in the construction.

Now we need to complete our proof obligations re the construction: we need to show that $M_{\alpha+2}$ is of cardinality μ.

coding functions defined; our strategy for counting $M_{\alpha+2}$: For each $x \in M_\alpha$ with strong support S, with strong support order $<_S$, define the coding function $\chi_{x,S}$ to satisfy the equation $\chi_{x,S}(<_{\rho[S]})) = \rho(x)$ for each α-allowable permutation ρ. This is well defined because $\rho[S] = \rho'[S]$ implies $\rho(x) = \rho'(x)$. The domain of $\chi_{x,S}$ is the "orbit" of the support order $<_S$ under suitable permutations in a natural sense. Our strategy for proving that there are no more than μ elements of $M_{\alpha+2}$ involves counting these "orbits" and then counting the possible coding functions. We show that there are $< \mu$ coding functions; there are exactly μ possible support orders (this is clear from how many atoms, local cardinals, and local cardinal elements there are) and so, since every object is obtained by applying a coding function to a support order, the result follows.

counting orbits in the strong support orders: We count the "orbits" in the strong support orders. To begin with, the orbits in $\nu+1$-supports are readily counted. A support order is an order on atoms in M_ν and near-litters in $M_{\nu+1}$, and the orbit is completely determined by specification of positions in the well-ordering $<_S$ at which there are atoms and positions in the well-ordering $<_S$ at which there are near-litters, and pairs of positions such that the atom in the first position is an element of the near-litter in the second position. It follows that there are $< 2^\kappa < \mu$ orbits in the $\nu+1$ supports. Coding functions acting on an orbit are determined as symmetric differences between the set of atoms at certain positions and the union (or the complement of the union) of the set of near-litters at certain positions. Again, there are $< 2^\kappa < \mu$ coding functions associated with each orbit.

The "orbits" in $\alpha+2$-support orders are counted using a more complex formal specification. The specification of an orbit is given by the following data:

1. The length of $<_S$.
2. For each ordinal position in the well-ordering $<_S$, the extended type index of the item there and the ordinal third component, and whether it is an atom, an element of a $K^a_{\beta+2}$, or a near-litter.
3. For each ordinal position occupied by an atom or an element of a $K^a_{\beta+2}$ paired with an extended type index A, the earlier ordinal position occupied by a local cardinal element containing it as an element or subset paired with A_1 or A. There will be just one such position by the disjointness criterion in the definition of support.
4. For each ordinal position occupied by a local cardinal element N with $[N]$ a litter element, paired with an extended type index A whose two smallest elements are successive, with the third component greater than the minimum of A, the earlier ordinal position occupied by a triple $([N], A_1, \gamma)$.
5. For each ordinal position occupied by a triple (N, A, δ) where the local cardinal element N has $[N]$ of the form $\sigma_{\gamma+2,\beta+2}(X)$ and the two smallest elements of A are $\beta+1, \beta+2 \leq \delta$, the $\gamma+2$-coding function which can be applied to the largest suborder of the segment in $<_S$ determined by (N, A) which determines a $\gamma+2$-strong support of X (with tweaks to indices removing A_1) to obtain X. The appropriate sublist can be determined simply by examining extended type indices.

6. For each ordinal position occupied by a triple (N, A, δ) where the local cardinal element N has $[N]$ of the form $\tau_{\beta+1,\gamma+3}(X)$ and the two smallest elements of A are $\gamma + 2, \beta + 2$, the $\beta + 1$-coding function which can be applied to the largest suborder of the segment in $<_S$ determined by (N, A) which determines a $\beta+1$-strong support of X (with tweaks to indices removing A_1) to obtain X. The appropriate sublist can be determined simply by examining extended type indices.

If two strong support orders $<_S$, $<_T$ have the same specification, we can show that there is ρ such that $\rho[S] = T$ (so they belong to the same orbit) by constructing a partial support map $[\rho]^0$ and so a family of locally small bijections ρ^0_A extensions of which will send $(x, A) \in S$ at a given position in $<_S$ to the (y, A) in the corresponding position in $<_T$. Where x and y are atoms or belong to a $K^a_{\beta+2}$, the ρ^0_A we construct maps x to y. Where x and y are local cardinal elements appearing with extended type index A, we have by inductive hypothesis already determined values of the desired bijection at a support for $[x]$ and so can determine that an extension of these values will send $[x]$ to $[y]$ by considering the coding function component of the orbit specification; we consider anomalies of x and y, elements of $x^+\Delta x^\circ$ and elements of $y^+\Delta y^\circ$ (x^+ being the near-litter to which x is equal or of which it is the set union): we extend $\rho^0_{A\cup\{\texttt{min}(A)-1\}}$ to map anomalies of x to elements or non-elements of y as appropriate and elements or non-elements of x as appropriate to anomalies of y, the new elements chosen to be distinct from atoms appearing in S and T with the same extended type index and not belonging to other near-litters appearing in S and T with the index A. This additional information ensures that x will be mapped to y by a ρ with ρ_A's extending ρ^0_A's without additional exceptions. This ensures that the orbit specifications do indeed specify orbits.

Observe that if there are known to be $< \mu$ β-coding functions for each $\beta < \alpha+2$, we can conclude that there are $< \mu$ orbits in the $\alpha+2$-support orders by counting these specifications, which are small structures built from extended type indices and coding functions of index $< \alpha + 2$.

counting coding functions, completing the argument that $M_{\alpha+2}$ is of size μ: Now we indicate how to specify the $\alpha+2$-coding functions in such a way as to make it clear that there are $< \mu$ of them (on the assumption that we have already shown this for $\beta < \alpha+2$). Select an x and a strong support S for x with support order $<_S$. For each $y \in x$, choose an $\alpha+1$-coding function $\chi_{y,T}$ where $<_T$ is obtained from an end extension of $<_S$ with suitable omissions of inappropriate elements and omission of $\alpha+2$ from extended type indices. We claim that the coding function $\chi_{x,S}$ is completely specified by the orbit specification of $<_S$ and the set of coding functions $\chi_{y,T}$, which establishes the desired result, since there are $< \mu$ such orbit specifications and the set of coding functions is a subset of a fixed collection of $< \mu$ coding functions and μ is strong limit. If we define $\chi^*_{x,S}(<_U)$, where $<_U$ has the same specification as $<_S$, as the collection of all $\chi_{y,T}(<_V)$, with $\chi_{y,T}$ one of the coding functions in the set chosen above and $<_V$ having the same specification as $<_T$ and being a modified end extension of $<_U$ in the correct sense, it is clear that $\chi_{x,S}$ is a coding function, and clear that $\chi^*_{x,S}(<_S)$ includes x as a subset, since it certainly includes each $\chi_{y,T}(<_T) = y$ for each $y \in x$. To complete the claim, we need to show that each $\chi_{y,T}(<_U)$ where $<_U$ is of the same specification as $<_T$ and is obtained from an end extension of $<_S$ in fact belongs to x. There is an allowable permutation ρ such that $\rho[T] = U$ and in addition the appropriate ρ_A fixes each element of S which is omitted from $<_U$. Observe that this ρ fixes S because it and its derivatives fix each element of S as appropriate, and in addition $\rho(y) = \rho(\chi_{y,T}(<_T)) = \chi_{y,T}(<_U)$, so the latter object is in x as desired.

We have now shown that the construction of M succeeds. We next address the status of images under the σ and τ maps which we have presumed were elements of M_α's.

elementwise images under σ maps in M_α's?: We observed above that for $A \in M_{\beta+3}$, $\bigcup(\sigma_{\beta+2,\alpha+2}``A) \subseteq M_{\alpha+1}$ and that it is demonstrable that this belongs to $M_{\alpha+2}$ with then unspecified exceptions. This does follow quite directly except in a special case: A has a $\beta+3$-indexed support because it is in $M_{\beta+3}$, and it is direct that the set obtained from this support by replacing each (s, B) with $(s, B \setminus \{\beta+3\} \cup \{\alpha+2\})$ is an $\alpha+2$-indexed support for this set – as long as the support of A does not contain any element with an extended type index component not including $\beta+2$; for example it cannot include set unions in $M_{\beta+3}$ of near-litters in $M_{\beta+4}$. This restriction on the construction of a "power set" of $M_{\beta+2}$ in $M_{\alpha+2}$ makes philosophical sense, as support elements in $M_{\beta+3}$ are off the sequence of types under consideration.

images under $f_{(0)}^{\alpha,\beta}$ in M_α's?: We observed that $f_0^{\alpha,\beta}(A) = \bigcup \sigma_{\beta+2,\alpha+2}``\bigcup(\tau_{\alpha+1,\beta+3}``A) \subseteq M_{\alpha+1}$ and we want to verify that this belongs to $M_{\alpha+2}$. We could chase this through the properties of the σ and τ maps, but it is much easier to observe that a defining property of allowable permutations is that they commute with $f_0^{\alpha,\beta}$ maps. so a support set is a support of $f_0^{\alpha,\beta}(A)$ (or $f^{\alpha,\beta}(A)$) if and only if it is a support of A.

5 M supports a model of TTT_λ

We now demonstrate that M supports a model of TTT_λ, in which type α for each $\alpha < \lambda$ is $M_{\alpha+2}$ and the relations $\in_{\beta,\alpha}$ are defined as indicated above. The approach we are taking here is basically that of a Fraenkel-Mostowski permutation method for proving independence of choice related assertions (for which we give [7] as a reference) but the situation here is sufficiently unfamiliar that it is necessary to prove the validity of our permutation method directly in the paper.

extensionality: That extensionality holds in M for each relation $\in_{\beta,\alpha}$ in a natural sense is evident from the definitions of these maps.

a basic observation: A straightforward calculation shows that

$$x \in_{\beta,\alpha} y \leftrightarrow \rho_{\beta+2,\alpha+2}(x) \in_{\beta,\alpha} \rho(y)$$

for any $\alpha + 2$-allowable ρ.

predicative tangled comprehension holds by permutation methods: From a formula ϕ of the language of TST obtain a formula ϕ^s by replacing each variable of type i with a variable restricted to $M_{s(i)+2}$, where s is a strictly increasing map from the finite set of types appearing in ϕ into λ, then translate each subformula $x^i \in y^{i+1}$ to $x \in_{s(i),s(i+1)} y$. Define A_i as the set of all $s(j)+2$ for $j \geq i$. We then observe that each formula $x \in_{s(i),s(i+1)} y$ is equivalent to $\rho_{A_i}(x) \in_{s(i),s(i+1)} \rho_{A_{i+1}}(y)$. From this we can conclude that a set $\{x : \phi^s\}$ is equivalent to its image under any permutation ρ_{A_i}, where i is the type of x, such that ρ_{A_j} fixes each parameter y in ϕ of type j. This looks like a result that $\{x : \phi^s\}$ has support with respect to allowable permutations, but only permutations ρ_{A_i} are considered and the highest type appearing in $\{x : \phi^s\}$ might be higher than the type of $\{x : \phi^s\}$. All difficulties are removed if we impose the restriction on ϕ that x be of the highest type that occurs in ϕ: we then find that ρ fixes $\{x : \phi^s\}$ just in case ρ_{A_j} fixes each parameter y of type j. Note also that as a set with no support elements of its own type, $\{x : \phi^s\}$ has all its elementwise images under σ maps belonging to M as well (it is "copied into higher type power sets").

we need the axiom of set union: Thus any set defined by an instance of the comprehension scheme of tangled type theory in M has support determined by the supports of its parameters, under the condition that the instance of comprehension is predicative: no free or bound variable in ϕ can be of higher type than the elements of the set being defined (though we do not need this, one can remark that parameters can be of the same type as the set being defined, as long as the parameters themselves do not have support elements of their own type). This gives us a model of tangled type theory of the species which yields consistency of Marcel Crabbé's theory NFP ("predicative NF", shown to be consistent in [1]). However, we can prove that the axiom of union holds in this model, and so that we get full tangled type theory and full NF. For any formula ϕ, existence of $\{\iota^j(x) : \phi\}$ for j large enough (ι denoting the singleton operation) will give a predicative instance of comprehension, and taking the set union j times will give the desired $\{x : \phi\}$.

proof that the axiom of set union holds: For this, it is sufficient to show that any subset of an $M_{\alpha+1}$ which has a γ-support with $\gamma > \alpha + 2$ actually has an $\alpha + 2$-support (obtained in the obvious way by dropping the elements of the γ-support which belong to M_γ's of too high an index and trimming indices). The argument above shows that any class (subset of an $M_{\alpha+1}$) obtained from an instance of the comprehension scheme of tangled type theory has such a support: its image under sufficiently many elementwise applications of appropriate singleton maps will have a support, which will also be a support for its sufficiently iterated set union; we can then cut this support down to a support with the right maximum to witness the fact that the class is actually a set in M.

Suppose that $X \subseteq M_{\alpha+1}$ has a γ-strong support S, with $\gamma > \alpha + 2$. Let S^- be obtained from S by replacing each $(s, B) \in S$ with $(s, B \cap \alpha + 3)$, and omitting any items obtained in this way which do not have $[\alpha + 3, \gamma] \subseteq B$. It should be clear that this process will produce a set eligible to be an $\alpha + 2$-strong support. We claim that S^- is in fact an $\alpha + 2$-strong support of X.

Let ρ be an $\alpha + 2$-allowable permutation such that $\rho[S^-] = S^-$: we need to prove that $\rho(X) = X$.

Choose $y \in X$. Let T be an $\alpha + 1$-support of y. Build a partial support map $[\chi]^0$ determining a γ-locally small family of bijections χ^0_A which agree with $\rho_{A\cap\alpha+3}$ where appropriate on elements of $S^- \cup T$ and fix all elements of S where the index is appropriate. The extension of the resulting family to a γ-allowable permutation χ is possible by our work above, and χ will fix X because it (or its appropriate derivatives) fixes all first components of elements of S and agree with ρ at y because it (or its appropriate derivative) has the same action as ρ on each first component of an element of the support T of y. But this implies that ρ maps X at least to a subset of X, and the same argument re ρ^{-1} shows that ρ fixes X, so X is an element of $M_{\alpha+2}$.

This completes the entire proof.

6 Conclusions and questions

The proof that NF is consistent, given above, will go through if $\lambda = \omega$ and $\kappa = \omega_1$.

If a typed assertion is true in all interpretations of TST_n in the model of TTT_λ for n large enough and satisfying enough Ambiguity, then the corresponding stratified assertion will be true in models of NF obtained from the construction. This is a way of investigating what facts might hold in such NF models; actual NF models are obtained by compactness.

Observe that any small subset of an M_α belongs to $M_{\alpha+1}$, with support obtained by taking the unions of supports of its elements all of whose (union of) near-litter elements are (unions of) litters. If one wants any mathematical structure of a known size to be well-orderable, choose κ larger than the size of that structure. The assertion that the reals can be well-ordered or the axiom of Dependent Choices will hold if we choose suitable values of

κ (the former if $\kappa > c$; the latter if $\kappa = \omega_1$) and so hold in the models of NF obtained from our construction; one can get relative consistency of stronger results of this kind by increasing κ. The axiom of Denumerable Choice which Rosser assumes in [10] holds with $\kappa = \omega_1$. This means that NF has no interesting (stratified) consequences in arithmetic or the theory of any familiar "small" mathematical structure describable in TST; choosing κ large enough ensures that the structure looks exactly the same in M as in the ground interpretation of ZFA, and looks the same in the models of NF obtained from the construction.

Relative consistency with NF of forms of the axiom of choice which don't involve a cardinality bound, such as the Prime Ideal Theorem or the assertion that the universe is linearly ordered, cannot be handled by our present methods.

There is a subtle point to be remembered: the model M contains the same small subsets of each M_α, and so certainly the same countable sets, as the ground interpretation of ZFA. It is not the case that an actual model of NF will contain even all of its countable subsets; but stratified combinatorial consequences for models of TST_n of existence in the model of all small sets of its non-top types which exist in the metatheory also hold in the models of NF obtained.

Using larger values of λ will allow proof of consistency of stronger extensions of NF, by using values of λ with stronger partition properties. The consistency of the Axiom of Cantorian Sets of Henson ([3]) or the Axioms of Small and Large Ordinals of this author (see [5], [12], [6]) can be established in essentially the same way that is reported in the author's paper [6] on strong axioms of infinity in NFU, assuming that λ is a suitable large cardinal and applying stronger partition properties than Ramsey's theorem.

An important point is that the existence of an ω-model of NF can be established. This can be done by brute force if one is willing to take λ a weakly compact cardinal, as one can then apply the argument to theories of models of TST_n expressed with infinitary conjunctions and disjunctions with $< \lambda$ terms, which yields models of NF with no nonstandard elements of λ. One can show the existence of an α-model for any fixed ordinal α, using considerably less consistency strength but more technical subtlety, by emulating Jensen's techniques in the original NFU consistency paper [8].

The existence of an ω-model of NF settles the old question of Maurice Boffa concerning the existence of an ω-model of TNT, the version of TST with sorts indexed by all integers, proposed by Hao Wang in [15]. An ω-model of NF immediately gives an ω-model of TST.

NOTE: this paragraph needs to be omitted or substantially rewritten, as it refers to a tangled web $\tau(A)$ which is not developed in this version of the proof. Internalizing the tangled web construction in tangled type theory could be briefly developed above. The existence of an ω-model also settles the esoteric question of whether the existence of cardinals with Specker trees of infinite rank is consistent with ordinary set theory. We explain this question and related known results briefly. The Specker tree of a cardinal μ has μ as its top node; each node is a cardinal and the children of a node ν are the preimages of ν under $(\kappa \mapsto 2^\kappa)$. Thomas Forster has shown, by refining an argument of Sierpinski (see [2], p. 48), that all Specker trees are well-founded, even in the absence of Choice, so every Specker tree has an ordinal rank in an obvious sense. Under Choice, the rank of every Specker tree is finite. In NF + Rosser's Axiom of Counting (an axiom originally proposed in [10] which holds in an ω-model) the cardinality of the universe can be proved to have infinite Specker rank. Up until now, it was unclear how one would construct a cardinal of infinite Specker rank in a set theory of the usual kind. If λ is uncountable, $\tau(A)$ for any A with an infinite minimum element has Specker tree of infinite rank in the FM interpretation that we exhibit above. This establishes consistency of existence of cardinals of infinite Specker rank with ZFA; we are confident that standard methods can be used to port this result to ZF.

We note that our paper thus shows the relative consistency of the system of Rosser's book [10], as we have indicated how to choose parameters to get his additional axioms (Denumerable Choice and his Axiom of Counting) to hold. We are happy about this because [10] is a lovely book about logic as the foundation of mathematics, which we commend to the reader, but it has been under a cloud since Specker's disproof of AC in NF in [13].

The results of this paper establish that NF is not very strong. We continue to believe that it is no stronger than TST with the Axiom of Infinity, which is of the same strength as Zermelo set theory with bounded separation. However, our results here do not establish this upper bound on the consistency strength of NF, as our argument in its present form requires the existence of a strong limit cardinal of cofinality ω_1.

This work does not answer the question as to whether NF proves the existence of infinitely many infinite cardinals (discussed in [2], p. 52). A model with only finitely many infinite cardinals would have to be constructed in a totally different way. We conjecture on the basis of our work here that

NF probably does prove the existence of infinitely many infinite cardinals, though without knowing what a proof will look like.

A natural general question which arises is, to what extent are *all* models of NF like the ones indirectly shown to exist here? Do any of the features of this construction reflect facts about the universe of NF which we have not yet proved as theorems, or are there quite different models of NF as well?

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