An intuitive overview of my construction of a tangled web

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This document is not a proof. It is intended to help a reader understand the intuition behind my argument for the consistency of NF.

It requires some familiarity with the actual papers presenting the proof (versions IIIa or IV on my web page are best).

The aim of the main construction is to build a tangled web of cardinals. We briefly review what this is:

We specify a limit ordinal λ. This may be ω, for the minimal aim of proving Con(NF).

Finite subsets of λ are called "clan indices". For any nonempty clan index A. A_1 is defined as $A \setminus \texttt{min}(A)$, A_0 is defined as A for any clan index A, and A_{n+1} is defined as $(A_n)_1$ for any A with at least $n+1$ elements.

$B \ll A$ is defined as holding if B is a proper superset of A and all elements of $B \setminus A$ are dominated by all elements of A.

Intuitively (way in the background) one should think of a clan index A as one of many indices for a type indexed by $\texttt{min}(A)$, or as an index for one of many copies of such a type.

A tangled web is a function τ from nonempty clan indices to (Scott) cardinals (existing in some model of ZFA without choice) with the following properties:

1. $2^{\tau(A)} = \tau(A_1)$, if $|A| > 1$ (naturality)
2. The theory of any natural model of TST_n with base type of size $\tau(A)$ is completely determined by $A \setminus A_n$ (the smallest n elements of A), if $|A| > n$ (elementarity).

We show in the paper that existence of a tangled web implies Con(NF).

We will start our work in a model of ZFA with choice, and define an FM interpretation of ZFA without choice in which a tangled web exists.

Let κ be a regular uncountable cardinal. We will refer to sets of cardinality $< \kappa$ as small and all other sets as large. It will be the case that all small subsets of the domain of our FM interpretation will be sets of the FM interpretation, so the meaning of "small" will not change. κ can be ω_1 for purposes of minimal results here.

Let μ be a strong limit cardinal which is greater than λ, greater than κ, and has cofinality at least κ. μ could be $\beth_{\omega_1}$ for purposes of minimal results here.

The atoms of our initial ZFA + Choice will include a pairwise disjoint collection of sets $\texttt{clan}[A]$, each of cardinality μ, which we will call "clans".

Let $\mathcal{P}_*$ denote the power set operation of the eventual FM interpretation, and let $|\cdot|_*$ denote (Scott) cardinality in the sense of the FM interpretation.

Our intention is that $\tau(A)$ will be $|\mathcal{P}_*^2(\texttt{clan}[A])|_*$ (this is what the clans are for). The group of permutations and filter on the group which we introduce are designed to enforce appropriate conditions on these cardinals.

Each clan is partitioned into μ sets of size κ called "litters". The collection of litters included in $\texttt{clan}[A]$ is called $\texttt{litters}[A]$.

There is a map Π, the global parent map, whose domain is the union of all the litter partitions $\texttt{litters}[A]$. $\Pi\lceil(\texttt{litters}[A])$ is a bijection from $\texttt{litters}[A]$ to a set of atoms $\texttt{parents}[A]$ (these are called parent sets), for each A. The set $\texttt{parents}[A]$ includes $\texttt{clan}[A_1]$ as a subset for each nonempty A. Each set $\texttt{parents}[A] \setminus \texttt{clan}[A_1]$ is of cardinality μ. For any A, B, $\texttt{parents}[A] \cap \texttt{parents}[B]$ is empty unless A, B are both nonempty and $A_1 = B_1$, in which case the intersection is $\texttt{clan}[A_1]$. The intersection of $\texttt{parents}[A]$ with $\texttt{clan}[B]$ is empty if $B \neq A_1$.

The sets $\texttt{clan}[A]$ and $\texttt{parents}[A]$ include all the atoms in our starting ZFA with choice. The atoms belonging to clans are called regular atoms; all others are called irregular atoms.

A "near-litter" is a subset of a clan with small symmetric difference from a litter. The collection of near-litters included in $\texttt{clan}[A]$ is called $\texttt{nearlitters}[A]$. The local cardinal of a litter L, written $[L]$, is the collection of near-litters with small symmetric difference from L. $\Pi(L)$ is called the "parent" of L and also the "parent" of each near-litter in $[L]$.

We will define for each nonempty A an injective map Π_A (the indexed parent map associated with A) whose domain is $\texttt{parents}[A] \setminus \texttt{clan}[A_1]$ and whose range is a collection of sets. We refer to $\Pi_A(\Pi(L))$ as the "set parent" of a litter L when it is defined. We will give much more specific information about Π_A presently.

We now have enough information to define our permutation group and filter (mod the exact definition of the maps Π_A).

Any permutation ρ acting on the atoms is taken to be extended to all sets by the convention $\rho(A) = \rho``A$.

An allowable permutation is defined as a permutation which satisfies the following three conditions:

1. ρ fixes each clan.
2. For any litter L, $\rho(L)$ is a near-litter with small symmetric difference from $\Pi^{-1}(\rho(\Pi(L)))$.
3. ρ fixes each map Π_A.

It is straightforward to show that an allowable permutation must fix each parent set; there is no harm in including this condition if one prefers to do so.

A support set is a small set of regular atoms and near-litters, with the property that any two near-litters belonging to the set are disjoint.

An object has support S iff S is a support set and any allowable permutation which fixes each element of S also fixes the object.

It is easy to see that any object which has a support has a support consisting entirely of atoms and litters. The reason that we allow support sets to contain near-litters is that this makes the image of a support set under an allowable permutation also a support set, which is technically extremely useful. At other points in the argument it is useful to restrict attention to supports which contain only litters and regular atoms.

A set is symmetric if it has a support, and hereditarily symmetric iff it is either an atom or all elements of its transitive closure are symmetric.

Standard results show that the collection of hereditarily symmetric objects is a class model of ZFA without choice. This is our FM interpretation (up to the specification of the Π_A's).

It is useful to have qualified versions of the symmetry concepts. Let B be a clan index.

A B-allowable permutation is defined as a permutation which satisfies the following three conditions:

1. ρ fixes each clan with index $\leqq B$.

2. For any litter L included in a clan with index $\leqq B$, $\rho(L)$ is a near-litter with small symmetric difference from $\Pi^{-1}(\rho(\Pi(L)))$.

3. ρ fixes each map Π_A with $A \ll B$.

A B-support set is a small set of regular atoms and near-litters belonging to or included in clans with index $\leqq B$, with the property that any two near-litters belonging to the set are disjoint.

An object has B-support S iff S is a B-support set and any B-allowable permutation which fixes each element of S also fixes the object.

We say something briefly here about intent. The idea is that allowable permutations will act quite freely on elements of litters: each litter will be κ-amorphous (it will have only small and co-small subsets in the FM interpretation) and the litters will all have distinct κ-amorphous cardinalities. The expectation is that the collection of subsets of $\texttt{clan}[A]$ which are the same size as a litter L included in that clan will be the local cardinal $[L]$ (thus the name "local cardinal"). Showing that the prerequisite freedom of action of allowable permutations actually holds is tricky. Further, the symmetric subsets of $\texttt{clan}[A]$ will be seen to be exactly the sets with small symmetric difference from small or co-small unions of litters: the clans are almost amorphous and in fact their power sets in the FM interpretation are all externally isomorphic to one another in terms of the ground interpretation of ZFA + Choice.

The map Π is not a set map in the FM interpretation, and $\texttt{litters}[A]$ is not a set in the FM interpretation.

Each of the maps Π_A is forced to be a set in the FM interpretation by the definition of allowable permutation (they are all invariant).

Observe that each local cardinal $[L]$ of a litter L is symmetric (with support $\{L\}$) and the map F sending each $\Pi(L)$ to $[L]$ is an invariant injective map, a set of the FM interpretation. Further, the map G_A sending any h.s. subset U of $\texttt{parents}[A]$ to $\bigcup\{F(u) : u \in U\}$ is an injective invariant map from $\mathcal{P}_*(\texttt{parents}[A])$ into $\mathcal{P}^2_*(\texttt{clan}[A])$, witnessing the cardinal inequality $|\mathcal{P}_*(\texttt{parents}[A])|_* \leq |\mathcal{P}^2_*(\texttt{clan}[A])|_*$. Further, observe that this implies that

$$|\mathcal{P}(\texttt{rng}(\Pi_A))|_* \leq |\mathcal{P}_*(\texttt{parents}[A])|_* \leq |\mathcal{P}^2_*(\texttt{clan}[A])|_*,$$

since the range of Π_A is the same size, in terms of the FM interpretation, as the domain of Π_A, which is a subset of $\texttt{parents}[A]$. [that each clan and parent set is symmetric and in fact invariant is easy to see].

This motivates our specification of the range of Π_A. In version IIIa (and all older versions of the construction) our stated intent is that

$$\texttt{rng}(\Pi_A) = \bigcup_{B \ll A} \mathcal{P}_*^{|B|-|A|+1}(\texttt{clan}[B])$$

(note that $|B| - |A| + 1$ is an integer indicating iteration of the symmetric power set operation).

It is by no means obvious that one can do this. It may look preposterous. What I don't say when I introduce this is why I am doing it. This definition of the range of the Π_A's serves to enforce the naturality conditions on the intended tangled web.

The naturality condition is $2^{\tau(A)} = \tau(A_1)$ for each A with $|A| > 1$. In terms of our intended definition of $\tau(A)$, this translates to

$$|\mathcal{P}_*^3(\texttt{clan}[A])|_* = |\mathcal{P}_*^2(\texttt{clan}[A_1])|_*.$$

We indicate why the stipulated ranges of the Π_A's will make this work, if they can be achieved.

Observe that $\texttt{parents}[A]$ includes $\texttt{clan}[A_1]$ and that $\texttt{parents}[A_1]$ includes $\Pi_{A_1}^{-1}``\mathcal{P}^{|A|-|A_1|+1}(\texttt{clan}[A])$, which is simply $\Pi_{A_1}^{-1}``\mathcal{P}^2(\texttt{clan}[A])$.

Thus we have

$$|\mathcal{P}_*^3(\texttt{clan}[A])|_* \geq |\mathcal{P}_*^2(\texttt{parents}[A])|_* \geq |\mathcal{P}_*^2(\texttt{clan}[A_1])|_*$$

and

$$|\mathcal{P}_*^2(\texttt{clan}[A_1])|_* \geq |\mathcal{P}_*(\texttt{parents}[A_1])|_* \geq |\mathcal{P}_*(\mathcal{P}_*^2(\texttt{clan}[A]))|_* = |\mathcal{P}_*^3(\texttt{clan}[A])|_*,$$

establishing the desired equation under the admittedly unlikely-looking hypotheses.

The signature feature of version IV is the observation that the more modest

$$\texttt{rng}(\Pi_A) = \bigcup_{\beta < \texttt{min}(A)} \mathcal{P}_*^2(\texttt{clan}[A \cup \{\beta\}])$$

has exactly the same effect: notice that the only inclusion of iterated power sets of clans in parent sets which is actually used in the argument above is the one enforced here.

Now we discuss the enforcement of the elementarity conditions. The elementarity conditions assert that natural models of TST_n with base type $\tau(A)$ and $\tau(B)$ have the same theory if $A \setminus A_n = B \setminus B_n$ (where $|A|, |B| > n$). We arrange for this to hold by enforcing the condition that the natural models in question are externally isomorphic. The index natural model of TST_n with base type of cardinality $\tau(A)$ is for us the model of TST_n with each type i implemented as $\mathcal{P}_*^{2+i}(\texttt{clan}[A])$ and with equality and membership implemented by the appropriate restrictions of the equality and membership relations of the ambient set theory.

We indicate the opening steps of enforcing the isomorphism conditions. For each $\alpha < \lambda$, choose a bijection σ_α whose domain is the union of all sets $\texttt{clan}[A]$ and $\texttt{parents}[A]$ with $\texttt{max}(A) < \alpha$. We extend σ_α to sets whose transitive closures contain no atoms not in its domain by the convention $\sigma_\alpha(A) = \sigma_\alpha``A$. Our further requirements are that the restriction of $\sigma_\alpha$ to each $\texttt{clan}[A]$ is a bijection from $\texttt{clan}[A]$ to $\texttt{clan}[\{\alpha\} \cup A]$, and for each $L \in \texttt{litters}[A]$, $\sigma_\alpha``L$ is a litter and $\sigma_\alpha(\Pi(L)) = \Pi(\sigma_\alpha``L)$.

Further, we require that $\sigma_\alpha(\Pi_A(\Pi(L)) = \Pi_{\{\alpha\}\cup A}(\sigma_\alpha(\Pi(L)))$, that is, that $\sigma_\alpha(\Pi_A) = \Pi_{\{\alpha\}\cup A}$. Notice that this requires us to enforce the condition that all atoms in the transitive closures of elements of $\texttt{rng}(\Pi_A)$ be in clans which are in the range of σ_α for any α dominating A, which will be true under either of the proposals for what the range of Π_A is, under both of which all atoms in its transitive closure belong to clans with index downward extending A.

We have set things up so that the maps σ_α must commute with all the relevant structure we have so far defined on our clans and parent sets (equality and membership, plus parents and set parents of litters). Note that this includes all the structure which specifies allowable permutations, supports and symmetry. The maps σ_α are not set maps in the FM interpretation but they are set maps in the ambient ZFA + choice. This suggests that for any $|A|, |B|$ with $A \setminus A_n = B \setminus B_n$ (and $n < |A|, |B|$) there will be a composition of sigma maps and inverses of sigma maps which will give an external isomorphism between the FM interpretation's natural models of TST_{n+2} with base types $\texttt{clan}[A]$ and $\texttt{clan}[B]$, and so between the FM interpretation's natural models of TST_n with base types $\mathcal{P}_*^2(\texttt{clan}[A])$ and $\mathcal{P}_*^2(\texttt{clan}[B])$, that is, natural models with base types of cardinalities $\tau(A)$ and $\tau(B)$, so of course these models will have the same theory extending

TST_n [this is not completely evident, since it might seem that elements of $\mathcal{P}^2_*(\texttt{clan}[A])$ and $\mathcal{P}^2_*(\texttt{clan}[B])$ might have essential support elements which are elements or subsets of clans with index not bounded by the indices of sigma maps used, which would mess this up: but we cite results below that show that in fact this cannot happen.]

Observe that "all we have to do" to specify Π_A for each A is specify $\Pi_{\{\alpha\}}$ for each $\alpha < \lambda$, in such a way as to enforce the peculiar specification of the range of this map. If $\Pi_{\{\alpha\}}$ can be defined with the correct range

$$\texttt{rng}(\Pi_{\{\alpha\}}) = \bigcup_{B \ll \{\alpha\}} \mathcal{P}_*^{|B|-|\{\alpha\}|+1}(\texttt{clan}[B]) = \bigcup_{B \ll \{\alpha\}} \mathcal{P}_*^{|B|}(\texttt{clan}[B])$$

or the more modest

$$\texttt{rng}(\Pi_{\{\alpha\}}) = \bigcup_{\beta<\alpha} \mathcal{P}^2_*(\texttt{clan}[\{\alpha,\beta\}])$$

then all instances of the formulas for ranges of Π_A's will be enforced by the properties of the sigma maps σ_α – if they work as external isomorphisms as we hope.

One obstruction is that we cannot actually try to specify the ranges of Π_A's as iterated symmetric power sets of clans, because we need to be acquainted with the Π_A's to know what the symmetry group is.

We resolve this problem by defining a stronger notion of symmetry: this is the major move which distinguishes version III from earlier versions (I *knew* about the strong symmetry property but had not thought of using it in the definition of the set parent maps). An element of $\mathcal{P}^{n+1}(\texttt{clan}[A])$ (where $|A| < n$) is *strongly symmetric* iff each of its elements is A_n-symmetric (the definition is actually more precise than this, introducing technicalities we do not discuss in this overview) and further each of its elements is also strongly symmetric (and so satisfies the stronger condition of A_{n-1}-symmetry, and so on, with further iterated members having stronger symmetry properties).

It is useful to note that the image under σ_α of the collection of strongly symmetric elements of $\mathcal{P}^{n+1}(\texttt{clan}[A])$ will be exactly the collection of strongly symmetric elements of $\mathcal{P}^{n+1}(\texttt{clan}[\{\alpha\}\cup A])$, if α dominates A, because of the way σ_α preserves relevant structure: the caveat expressed above about the possible action of σ_α's as external isomorphisms between iterated symmetric power sets does not apply here, because the indices of clans in which support elements are subsets or elements are forced to be suitably bounded by the definition of strong symmetry.

It is reasonably straightforward to verify that knowledge of B-symmetry for any B requires information only about Π_C's for $C \ll B$, which reduces to knowledge of $\Pi_{\{\mathtt{min}(C)\}}$ where $\mathtt{min}(C) < \mathtt{min}(B)$, so knowledge of strong symmetry of an element of $\mathcal{P}^{n+1}(\mathtt{clan}[A])$ depends only on knowledge of $\Pi_{\{\beta\}}$ for $\beta < \mathtt{min}(A_n)$.

Mod additional technicalities, we will define $\Pi_{\{\alpha\}}$ as a bijection (in the sense of the ambient ZFA with choice; this will be mostly invisible to the FM interpretation!) from $\mathtt{parents}[\{\alpha\}] \setminus \mathtt{clan}[\emptyset]$ to the collection of strongly symmetric elements of $\bigcup_{B \ll \{\alpha\}} \mathcal{P}^{|B|}(\mathtt{clan}[B])$ or, more modestly, the collection of strongly symmetric elements of $\bigcup_{\beta<\alpha} \mathcal{P}^2(\mathtt{clan}\{\alpha, \beta\}])$. It is straightforward to compute that one needs information only about $\Pi_{\{\beta\}}$'s for $\beta < \alpha$ to specify either of these sets. The additional technicalities have to do with choosing the bijection *very carefully* to get desired technical properties of supports. Once this is done, it follows by the properties of the sigma maps that each Π_A is a bijection from $\mathtt{parents}[A] \setminus \mathtt{clan}[A_1]$ to the strongly symmetric elements of $\bigcup_{B \ll A} \mathcal{P}^{|B|-|A|+1}(\mathtt{clan}[B])$ (or of the smaller union of iterated power sets specified in version IV).

There is an obvious proof obligation here: we need to show that the collection of strongly symmetric elements of $\bigcup_{B \ll \{\alpha\}} \mathcal{P}^{|B|}(\mathtt{clan}[B])$ is of the same cardinality μ as $\mathtt{parents}[\{\alpha\}] \setminus \mathtt{clan}[\emptyset]$ in the ambient ZFA with choice. This can be shown by showing that the collection of strongly symmetric elements of any $\mathcal{P}^{n+1}(\mathtt{clan}[A])$ ($n < |A|$) is of cardinality μ. This is done in the paper by a careful analysis of orbits in the allowable permutations: the allowable permutations act in a suitable sense very freely, and there are relatively few hereditarily symmetric sets.

The further subtler proof obligation is to show that the hereditarily symmetric elements of $\mathcal{P}^{n+1}(\mathtt{clan}[A])$, for $n < |A|$, are exactly the strongly symmetric elements: they clearly include the strongly symmetric elements, but it is not obvious that there are no hereditarily symmetric elements which are not strongly symmetric. This is again proved via a careful analysis of the freedom of action of allowable permutations. This has two uses: it verifies that the sets chosen as ranges of the Π_A's actually meet the intended specification as unions of iterated symmetric power sets, and it assists the proof of elementarity (removing the caveat expressed above).

As soon as these two things are shown, the translations of the naturality and elementarity conditions are seen to hold and our aims are met.