Quine's set theory "New Foundations" is consistent

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6/14/2017 3 pm Boise time

1 Version Remarks

6/14/2017 Tidied up the description of the inductive argument for coincidence of notions of symmetry. X is an *element* not a subset of $\mathcal{P}^{n+1}(\texttt{clan}[A])$ in that argument.

6/10/2017 corrected the application of the induction in the extension property proof to the better form in the last submitted version. But the overall format of this version is much better.

6/9/2017 additional note: it appears that the proof of the extension property in `submissionalt-with-a-repair` does not involve the same error that I fixed in the proof of the extension property here; I applied the induction hypothesis in a different way. But I think the argument here is clearer, once fixed. 4:30 remarks about relations to the previous version, no changes to text.

6/9/2017 A complete proofreading pass. In the section on coincidence of notions of symmetry, refined the instructions on how ρ_0' is constructed. The 5:30 correction yesterday removed the need for complex reasoning to justify $<_T$ being an end extension of $<_S$ in the counting orbits proof.

6/8/2017 Some proofreading. 7:45 am. More serious work on the counting orbits proof and the section of technical results about supports. Some attention was needed to showing that $<_T$ really can be arranged to be an end extension of $<_S$ in the counting orbits proof. There are also a lot of quite annoying local typos in the proof, some of which would be quite puzzling for a reader. I'm working on cleaning them out. There is probably a need for parallel debugging in the counting orbits proof in the latest submitted version with repairs. 5:30 pm significant rephrasing of what is proved in the counting orbits argument (the 5:30 fix it brings it into line with older versions in a particular respect, no corrections needed elsewhere).

I've spent a lot of time unfolding dense lists of conditions into enumerated lists of points. I hope this helps.

6/7/2017 Improved the writing in section 4 (technical results about supports). Further work on the induction in the proof of the extension property: this requires extreme care, but I think I finally have it down. This repair induces a requirement for repair in the last submitted version as well. 7:15 pm a further fix to the proof of the extension property: it is converging, but there is something fairly nasty that I didn't notice in the original argument for this version. Note added later: it appears that I did not make the same error in the previous version, though I think the approach taken here (once corrected) is clearer.

6/5/2017 cleared old version remarks. Corrected an error in the 6/4 edits. Corrected a seriously stupid typo in the presentation of the older version of the construction of the Π_A's on page 8: that is "the set of strongly symmetric D for which there is B with $|B| \geq 2$ and $\texttt{max}(B) = \alpha \ldots$ ", not $\texttt{min}(B) < \alpha$. Strengthened the definition of "strong support" to include one of the properties up until now assigned to extended strong supports. Firmed up the description of the induction in the proof of the extension property.

The organization of this version is quite different, though the argument is basically the same as in the last version of the submitted paper. In this version, I simply dive right into the construction (without foreshadowing or much attempt at motivation), do everything step by step and say how it shows consistency of NF at the end (introducing NF considerations briefly exactly when needed).

The closure conditions on strong supports in this version are stronger; this makes it easier to write arguments by induction along support orders in certain cases.

I discovered one notable error in the latest submitted version which I repaired in the "version with repairs" of that paper as well as here; the theory of the natural model of TST_{n+2} with base type $\texttt{clan}[A]$ depends on the $n+1$ smallest elements of A, not on the n smallest elements. This was annoying because I had noticed and fixed this in the last generation of edits before submission: I had the mistaken idea that changes in the argument just before submission of the paper had changed the situation. The effects of this are entirely local to the last part of the proof: one carries out the final

Ramsey's theorem application using partitions of $n+1$ element subsets of λ rather than n element subsets, and it all works.

I have made some improvements in notation and terminology, but it is mostly similar to that of other recent drafts. Notable new notions are $\texttt{setparents}[A]$, and the flavors "orderly" and "overspecified" of supports.

2 Introduction

We show that the set theory New Foundations (NF), so-called after the title of Quine's 1937 paper [8] introducing it, is consistent. NF considerations are not mentioned until late in the paper.

We work in ZFA (the usual set theory with atoms, including choice). We will call the ambient ZFA the ground interpretation: we will be defining a Fraenkel-Mostowski (FM) interpretation of ZFA without choice with which the ground interpretation will be contrasted. The fact that an FM construction is used should not be surprising, in the light of the scandal revealed in Specker's [12]: NF disproves the Axiom of Choice.

We will describe the atoms below.

3 The construction

Fix a limit ordinal λ. The value $\lambda = \omega$ would suffice for simply proving Con(NF).

Definition: A *clan index* is defined as a finite subset of λ. If A is a nonempty clan index, A_1 is defined as $A \setminus \{\texttt{min}(A)\}$. A_0 is defined as A for any clan index A; A_{n+1} is defined as $(A_n)_1$ where A is a clan index with at least $n+1$ elements. Where A, B are clan indices, we define $A \ll B$ as meaning "A is a proper subset of B, and each element of $B \setminus A$ is less than each element of A"; in English we may say "A properly downward extends B". $A \underline{\ll} B$ means $A \ll B \vee A = B$, and may be read "A downward extends B".

Fix a regular uncountable cardinal κ. The value $\kappa = \omega_1$ would suffice for simply proving Con(NF).

Definition: A set is said to be *small* iff its cardinality is less than κ. Sets which are not small are *large*.

Fix a strong limit cardinal μ such that $\mu > \kappa$, $\mu > \lambda$, and the cofinality of μ is $\geq \kappa$.

The atoms are postulated: There are μ atoms. The atoms are partitioned into the following sets, each of size μ: a set $\texttt{parents}[\emptyset]$; for each clan index A a set $\texttt{clan}[A]$; for each nonempty clan index A a set $\texttt{setparents}[A]$[1]. We reiterate that each of these sets is of cardinality μ, no two of them intersect, and they exhaust the atoms. The atoms belonging to sets $\texttt{clan}[A]$ are referred to as "regular atoms" and the others are referred to as "irregular atoms". For each nonempty clan index A, $\texttt{parents}[A]$ is defined as $\texttt{clan}[A_1] \cup \texttt{setparents}[A]$. The sets $\texttt{clan}[A]$ are referred to as "clans" and the sets $\texttt{parents}[A]$ are referred to as "parent sets". The index of an atom belonging to $\texttt{clan}[A]$ or $\texttt{parents}[A]$ is defined as A.

The litter partitions are postulated: Each clan $\texttt{clan}[A]$ is supplied with a fixed partition $\texttt{litters}[A]$ into sets of cardinality κ. Note that the partitions will be of size μ. These partitions are called "litter partitions". A set which belongs to a litter partition is called a "litter". A subset of a clan with small symmetric difference from a litter is called a "near-litter". The set of near-litters included in $\texttt{clan}[A]$ is called $\texttt{nearlitters}[A]$. For any near-litter N, we let N° denote the unique litter with small symmetric difference from N. We refer to elements of $N \Delta N^\circ$ as *anomalies* of N. The index of a near-litter belonging to $\texttt{nearlitters}[A]$ is defined as A.

The (atomic) parent map is postulated: We fix a map Π whose domain is the union of all litter partitions and whose range is the union of all parent sets, such that the restriction of Π to each $\texttt{litters}[A]$ is a bijection from $\texttt{litters}[A]$ to $\texttt{parents}[A]$.[2] The *(atomic) parent* of any near-litter N is $\Pi(N^\circ)$.

The set parent maps are introduced: We will provide for each nonempty clan index A a bijection Π_A from $\texttt{setparents}[A]$ to a collection of sets. The only stipulation we make about the ranges of the Π_A's for the moment is that for any clan index A, if an atom is in the transitive

[1] This is a new notation, but the same structure is being defined: taking $\texttt{setparents}[A]$ as basic rather than $\texttt{parents}[A]$ simplifies the description a bit.

[2] Since distinct sets $\texttt{parents}[A]$ have nonempty intersection, Π itself is not injective.

closure of the range of Π_A, it will belong to some $\texttt{clan}[B]$ with $B \ll A$. Other details of these maps will be given later. For any near-litter N for which $\Pi_A(\Pi(N^\circ))$ exists, we call this the *set parent* of N (the image of the atomic parent under Π_A). Recall that some elements of $\texttt{nearlitters}[A]$ (for nonempty A) have atomic parent in $\texttt{clan}[A_1]$ and so do not have set parents.

We now have enough information to start describing the group and filter determining the FM interpretation.

Convention: Any map ρ from a set of atoms to a set of atoms is extended to sets whose transitive closures contain no atom not in the domain of ρ by the rule $\rho(A) = \rho``A$.

Definition: For each clan index A, an *A-allowable permutation* is a permutation ρ of the atoms which

1. fixes each clan $\texttt{clan}[B]$ where $B \underline{\ll} A$,
2. sends any litter with parent p belonging to a clan $\texttt{clan}[B]$ where $B \underline{\ll} A$ to a near-litter with parent $\rho(p)$,
3. and fixes each set parent map Π_A, where $B \ll A$. Note that this implies that a litter L with set parent X included in such a $\texttt{clan}[B]$ is mapped to a near-litter $\rho(L)$ with set parent $\rho(X)$.

We refer to a $\emptyset$-allowable permutation as simply an allowable permutation.

Observations: Notice that it is straightforward to prove that $\texttt{parents}[B]$ is fixed by an A-allowable permutation if $B \underline{\ll} A$.

Notice that to determine whether a permutation is A-allowable requires no information about any Π_B unless $B \ll A$.

Notice that a B-allowable permutation is A-allowable if $A \ll B$, and in particular any allowable permutation is A-allowable for any A.

Definition: For each clan index A, an A-support set is a small set S of atoms and near-litters belonging to sets $\texttt{clan}[B]$ and $\texttt{litters}[B]$ with $B \underline{\ll} A$, with distinct near-litter elements of S disjoint from one another. A $\emptyset$-support set is simply referred to as a support set.

An object x is said to have *A-support S* iff S is an A-support set and each A-allowable permutation ρ such that $(\forall s \in S : \rho(s) = s)$ fixes x. If $A = \emptyset$, we simply say that x has *support S*.

An object is said to be *A-symmetric* if it has an A-support, and simply *symmetric* if it has an $\emptyset$-support, and further *(A-)hereditarily symmetric* if it is an atom or if it is symmetric and every element of its transitive closure is (A-)symmetric.

Observation: Note that any object with an A-support has an A-support all of whose near-litter elements are litters: this is readily obtained by replacing each near-litter N appearing in the A-support with the litter N° and the atoms which are anomalies of N. That is, if S is an A-support of the object x, then

$$S^\circ = \{N^\circ : (\exists B : N \in \texttt{nearlitters}[B] \cap S)\}$$

$$\cup \bigcup \{N \Delta N^\circ : (\exists B : N \in \texttt{nearlitters}[B] \cap S)\}$$

$$\cup \{x : (\exists B : x \in \texttt{clan}[B] \cap S)\}$$

is also an A-support for x, containing only atoms and litters.

Notice that an A-support set is a B-support set where $A \ll B$, and any A-(hereditarily) symmetric set is B-(hereditarily) symmetric for any $A \ll B$, and in particular that an A-support of an object is a support of the object and an A-symmetric set is symmetric.

We allow support sets to contain near-litters as well as litters because it is technically convenient (indeed very important for arguments to be tolerable) that the image of a support set under an allowable permutation be a support set (more generally, the image of an A-support set under a B-allowable permutation is an A-support set, if $A \lll B$, and if ρ is a B-allowable permutation ($A \lll B$) and S is an A-support for x, then $\rho(S)$ is an A-support for $\rho(x)$).

Entirely standard considerations show that the atoms and hereditarily symmetric sets of the ground interpretation make up an FM interpretation of ZFA, presumably without choice. We cannot tell much about it until we know more about the maps Π_A.

The group and filter used to define the FM interpretation are the group G of allowable permutations and the filter Γ of subgroups G_S (S a support set) consisting of the allowable permutations ρ such that $(\forall s \in S : \rho(s) = s)$. An example of the usefulness of allowing near-litters in support sets is that it makes it very direct to see that the filter is normal. See [6] for details of how an FM interpretation is defined in terms of the group and the filter.

We now postulate structural correspondences between clans whose indices have the same smallest element, which we will use to guide the construction of the maps Π_A which completes the description of the structure we are working on.

structural correspondence maps postulated: For each ordinal $\alpha < \lambda$, we fix a map σ_α acting on all atoms in clans $\texttt{clan}[A]$ and parent sets $\texttt{parents}[A]$ such that $\texttt{max}(A) < \alpha$, and extended to sets following our convention given above. The restriction of σ_α to each $\texttt{clan}[A]$ with $\texttt{max}(A) < \alpha$ is a bijection to $\texttt{clan}[A \cup \{\alpha\}]$. The restriction of σ_α to each $\texttt{litters}[A]$ with $\texttt{max}(A) < \alpha$ is a bijection to $\texttt{litters}[A \cup \{\alpha\}]$. We stipulate that $\sigma_\alpha(\Pi(L)) = \Pi(\sigma_\alpha(L))$ for each appropriate litter L. We further stipulate that $\sigma_\alpha(\Pi_A) = \Pi_{A\cup\{\alpha\}}$ where $\texttt{max}(A) < \alpha$ [we have already stipulated conditions that ensure that Π_A lies in the domain of the action of σ_α extended to sets]. The stated conditions on clans and the atomic parent maps can certainly be satisfied (this is just an issue of cardinalities). The condition on the Π_A's can be met by first choosing each $\Pi_{\{\alpha\}}$ for $\alpha < \lambda$ in some way yet to be designated (this will be specified below), then computing each Π_A by application of the appropriate finite composition of sigma maps to $\Pi_{\{\texttt{min}(A)\}}$.

We now describe some refinements of the notion of support.

Definition: An *A-strong support set* is a A-support set S equipped with a well-ordering $<_S$ such that[3]

1. For each atom x in S, a near-litter containing the atom x appears earlier in the order $<_S$ than x (there will be exactly one of these because distinct near-litters in a support set are disjoint).

[3]It may be useful to the reader to notice that we impose stronger closure conditions on strong supports here than in other recent versions of the argument. This simplifies the argument in various places. It does mean that reasons for calling an extended strong support "extended" have largely disappeared, as the merely strong supports have essentially the same closure conditions.

2. For each near-litter N in S, if it has no set parent and belongs to a $\texttt{nearlitters}[B]$ with $B \ll A$, its atomic parent (in $\texttt{clan}[B_1]$) appears before N in the order $<_S$.

3. For each near-litter N in S belonging to $\texttt{nearlitters}[B]$ with $B \ll A$ and having set parent, all elements of an B-(strong)[4] support of the set parent of N appear before N in the order $<_S$.

An object x has strong support S iff S is a strong support set and x has support S.

An A-*extended strong support set* is an A-strong support set S with associated order $<_S$ with the further property that each near-litter belonging to it is a litter. An A-*overextended strong support set* is an A-extended strong support set with the further property that for each near-litter N in S belonging to $\texttt{litters}[A]$ and having set parent, all elements of an A-(overextended strong) support of the set parent of N appear before N in the order $<_S$. An object x has A-(over)extended strong support S iff S is an (over)-extended strong support set and x has support S. Note that an (over)-extended strong support may from time to time be called just an (over)extended support.

Convention continued: As before, we feel free to drop the qualification of an object or property with a clan index if the clan index is empty.

Observations: Note that identifying an A-extended strong support set requires information about maps Π_B only for $B \ll A$. Identifying A-overextended strong support sets requires knowledge of Π_A.

Notice that if ρ is an A-allowable permutation and S is an A-strong support set with support order $<_S$, then $\rho(S)$ is an A-strong support set with support order $\rho(<_S)$. This is not true for A-extended or A-overextended supports because an allowable permutation may send litters to near-litters which are not litters.

It is worth noting that in an A-extended strong support S, the B-support of the set parent of a litter in $\texttt{litters}[B]$ belonging to S, where $B \ll A$, which is included in the $<_S$-segment before the litter may be taken to be a B-*over*extended strong support.

[4]Note specifically that this is a condition originally imposed on extended strong supports: originally I just had A-support here.

We now define a special notion of symmetry useful in defining the maps Π_A.

Definition: An element of an iterated power set $\mathcal{P}^{n+1}(\texttt{clan}[A])$, where $n < |A|$, is said to be *strongly symmetric* iff all of its elements are atoms or strongly symmetric and it has an A_n-extended strong support. This is of course a definition by recursion on n.

Observation: Note that any set which belongs to some set $\mathcal{P}^{n+1}(\texttt{clan}[A])$ belongs to exactly one such set unless it is a hereditarily finite pure set, and that hereditarily finite pure sets are A-symmetric with empty support for any A in any case, and so strongly symmetric.

Notice that a strongly symmetric set is hereditarily symmetric.

Notice that the elements of a strongly symmetric element of $\mathcal{P}^{n+1}(\texttt{clan}[A])$ (if they are not atoms) are A_{n-1}-symmetric, that is symmetric in a stronger sense.

Notice that the image under a map σ_α of a set X belonging to $\mathcal{P}^{n+1}(\texttt{clan}[A])$ (α dominating A, $n < |A|$) will be strongly symmetric iff X is strongly symmetric. It is quite evident that if $x \in \mathcal{P}^{n+1}(\texttt{clan}[A])$ has A_n-support S, then $\sigma_\alpha(x)$ has $(A_n \cup \{\alpha\}) = (A \cup \{\alpha\})_n$-support $\sigma_\alpha(S)$, and vice versa.

We now describe the construction of maps $\Pi_{\{\alpha\}}$, recalling that the stipulated properties of the structural correspondence maps ensure that we know how to compute Π_A once we have computed $\Pi_{\{\texttt{min}(A)\}}$, and assuming that we have already successfully defined $\Pi_{\{\beta\}}$ for each $\beta < \alpha$.

Construction of $\Pi_{\{\alpha\}}$: For an $\alpha < \lambda$, suppose that we have constructed all $\Pi_{\{\beta\}}$ for $\beta < \alpha$.

Let X_α be the set of all strongly symmetric sets D for which there is a $\beta < \alpha$ such that $D \in \mathcal{P}^2(\texttt{clan}[\{\alpha, \beta\}])$ [the older definition of X_α is as the set of strongly symmetric D for which there is B with $|B| \geq 2$ and $\texttt{max}(B) = \alpha$ (equivalently, $B \ll \{\alpha\}$) such that $D \in \mathcal{P}^{|B|}(\texttt{clan}[B])$; we will provide the complete argument for the older definition as well.] To do this (for either definition) requires information about maps Π_C for $C \ll \{\alpha\}$, which we can determine from knowledge of $\Pi_{\{\texttt{min}(C)\}}$, which we have by inductive hypothesis, since $\texttt{min}(C) < \alpha$.

If X_α is of cardinality $> \mu$, the construction fails and we stop [we will prove below that this cannot happen.] That the cardinality of X_α is at least μ is obvious (consider iterated singletons of atoms).

Otherwise, we stipulate that the range of $\Pi_{\{\alpha\}}$ will be X_α. We select a well-ordering $<^1_{\{\alpha\}}$ of $\texttt{parents}[\{\alpha\}]$ of order type μ, with the property that objects in positions with ordinal index with even finite part are in $\texttt{clan}[\emptyset]$, and a well-ordering $<^2_{\{\alpha\}}$ of order type μ of X_α, and let $\Pi_{\{\alpha\}}$ be a bijection from $\texttt{setparents}[A]$ to X_α with a technical property important for support structure: for each x in $\texttt{setparents}[\{\alpha\}]$, $\Pi_{\{\alpha\}}(x)$ has an $\{\alpha\}$-extended support which contains no near-litter with parent an element y of $\texttt{parents}[\{\alpha\}]$ with y not $<^1_{\{\alpha\}} x$. We achieve this straightforwardly: choose images under Π_A in stages indexed by ordinals $< \mu$: at stage γ, if the γth element of the order $<^1_{\{\alpha\}}$ on $\texttt{parents}[A]$ is an element of $\texttt{clan}[\emptyset]$, do nothing; if the γth element of $\texttt{parents}[A]$ is an element of $\texttt{setparents}[A]$, map it to the first element of X_α in the order $<^2_{\{\alpha\}}$ which has not already been chosen as an image under $\Pi_{\{\alpha\}}$ and which has an $\{\alpha\}$-extended strong support containing no near-litter with parent an element of $\texttt{parents}[\{\alpha\}]$ with index greater than or equal to γ in $<^1_{\{\alpha\}}$. The facts that support sets are small and that the cofinality of μ is at least κ ensure that the map will be onto. The condition that things with even index in the order on $\texttt{parents}[\{\alpha\}]$ are in $\texttt{clan}[\emptyset]$ prevents this from failing early for silly reasons.

This completes the description of the construction of the Π_A's. The obligation remains to show that the cardinality of each set X_α is μ, so that all stages of the construction succeed.

Note from above that as soon as we have defined $\Pi_{\{\alpha\}}$ we are able to define Π_A for any A where $\texttt{min}(A) = \alpha$.

It is convenient to further define orders $<^i_{B\cup\{\alpha\}}$ where $i = 1, 2$ and B is a clan index dominated by α as $\sigma_\alpha(<^i_B)$. Commutativity of relevant structure with the structural correspondence maps then ensures that for each clan index A for which Π_A is successfully constructed, for each x in $\texttt{setparents}[A]$, $\Pi_A(x)$ has an A-extended support which contains no near-litter with parent an element y of $\texttt{parents}[A]$ with y not $<^1_A x$.

Definition: We define the notion of *A-orderly support* for set parents of litters in $\texttt{litters}[A]$. If L is a litter belonging to $\texttt{litters}[A]$ with parent x and set parent $\Pi_A(x)$, an A-orderly support for $\Pi_A(x)$ is an A-extended support S for $\Pi_A(x)$ containing no near-litter included in $\mathtt{clan}[A]$ with parent $y \not<^1_A x$ (such a support always exists by the construction).

We further note that the range of each Π_A will be the collection of strongly symmetric sets in $\bigcup_{\beta<\mathtt{min}(A)} \mathcal{P}^2(\mathtt{clan}[A \cup \{\beta\}])$, or under the older definition of X_α, the collection of strongly symmetric sets in the forbidding

$$\bigcup_{B \ll A} \mathcal{P}^{|B|-|A|+1}(\mathtt{clan}[B]).$$

The account of the construction is now complete. We now prove that it has desired properties. Of crucial importance is to prove that Π_A is actually defined for every A (for which it suffices to show that X_α is of size μ for each $\alpha < \lambda$).

4 Technicalities about strong supports

In this section, we prove some basic facts about strong supports.

Merging extended strong supports: Given any small collection Σ of extended strong support sets (in which all near-litter elements are litters) it is obvious that the union of the collection is a support set and fairly easy to show that it is an extended support set. To show the latter fact, we need to construct an order on $\bigcup \Sigma$. To do this, place any desired well-ordering $<_1$ on Σ: use this to define a well-ordering $<_2$ on the disjoint union of Σ, the set of all $\{(x, S) : x \in S \wedge S \in \Sigma\}$: $(x, S) <_2 (y, T)$ iff $(S = T \wedge x <_S y) \vee S <_1 T$. Now define the order $<_3$ on the union of Σ as follows: $x <_3 y$ iff

$$(\exists ST : (x, S) <_2 (y, T) \wedge (\forall U : \neg((x, U) <_2 (x, S) \vee (y, U) <_2 (y, T)))).$$

In English, to get the support order on $\bigcup \Sigma$, concatenate copies of the support orders on the elements of Σ in some arbitrary order, then eliminate redundant copies of items belonging to more than one of the supports, always preserving the copy appearing earliest.

It is straightforward to see that this order is an extended strong support order.

How to merge a small collection of strong supports: We indicate how to merge any small collection of orders on strong support sets (which may contain near-litters which are not litters) into an order witnessing the fact that the union of the collection (somewhat modified) is a strong support set. All these remarks apply to A-strong supports for any given A; we are simply eliding the index for convenience.

Let Σ be a collection of strong support orders $<_{S_\alpha}$ on which we have supplied an ordinal indexing. Initially, define $x <_{S'} y$ for $x, y \in \bigcup_\alpha S_\alpha$ as $(\exists\alpha : x \in S_\alpha \wedge (\forall\beta < \alpha : x \notin S_\beta \wedge y \notin S_\beta) \wedge (x <_{S_\alpha} y \vee (y \notin S_\alpha \wedge (\exists\gamma : y \in S_\gamma)))$ It is necessary to correct for the possibility that distinct near-litters which intersect may appear in this preliminary order. Where a near-litter L appears before a near-litter M with small symmetric difference from it, replace L with $L \cup M$ in the same place in the order, delete M from the order, and add any atoms in $L \Delta M$ which are not already in the domain of the order as a block in the former position of M. Where a litter L appears before a litter M with large symmetric difference from it, replace M with $M \setminus L$ in the former position of M and insert the atoms in $L \cap M$ which do not already belong to the domain of the order as a block right after the former position of M. These modifications, carried out along the order $<_{S'}$ will not perturb the other conditions on a strong support and will enforce the condition that distinct near-litters in the domain of the order are disjoint. Whenever a support set is modified in this way, any object which had the original support set as a support also has the modified support set as a support.[5]

Every support element of index B has a B-extended strong support: We first observe that any litter included in $\texttt{clan}[B]$ has an extended strong support. If it has parent in $\texttt{clan}[B_1]$, its singleton is a B-extended strong support for it. If it has set parent, the set parent has a B-extended strong support by the construction: appending the litter to the support order for the set parent gives the order on a B-extended support for the litter. Append an atom in $\texttt{clan}[B]$ to the support order for a B-extended support for the litter containing the atom to get the support order for a B-extended support of the atom. A B-extended support order for a near-litter N can be obtained by merging a B-extended support for N° with B-extended supports for

[5]It is not clear that we use this anywhere, but it is useful to observe that it is possible.

the anomalies of N using the method described above.

Every support element with index B has B-overextended strong support: We argue by induction on the order $<^1_B$ that every set parent of an element of $\texttt{litters}[B]$ has a B-orderly overextended support. For each x in $\texttt{setparents}[B]$, $\Pi_B(x)$ has a B-extended support which contains no near-litter with parent an element y of $\texttt{parents}[B]$ with y not $<^1_B x$ by the construction (i.e, a B-orderly support): merge the order on this support with the orders on the B-orderly overextended supports of each $\Pi_B(y)$ for which $(\Pi\lceil\texttt{litters}[B])^{-1}(y)$ appears in the support ($y <^1_B x$ by orderliness) which exist by inductive hypothesis, using the method given above for merging B-extended support orders. The resulting order will witness the fact that its domain is a B-orderly overextended support for $\Pi_B(x)$. This shows that the litter $(\Pi\lceil\texttt{litters}[B])^{-1}(x)$ has a B-overextended support, the order on which is obtained by appending $(\Pi\lceil\texttt{litters}[B])^{-1}(x)$ to the previously obtained order. A litter L in $\texttt{litters}[B]$ with parent in $\texttt{clan}[B_1]$ has B-overextended support $\{L\}$. All atoms in $\texttt{clan}[B]$ can be seen to have B-overextended strong supports since the litters to which they belong have been shown to have B-overextended strong supports. Similarly, a near-litter with index B has B-overextended strong support obtainable by merging such a support for the litter from which it has small symmetric difference with such supports for its anomalies.

All objects with index $\underline{\ll} A$ have A-extended strong support : The method of construction of the set parent maps ensures that if we have defined all Π_B for $B \ll A$, it follows that any object in a clan or parent set with index $\underline{\ll} A$ has an A-extended support. Begin with any support for the object. Replace each near-litter N in it which is not a litter with the litter N° immediately followed by any anomalies of N which do not appear elsewhere in the order already. Merge the modified support with B-overextended strong supports for each atom in a $\texttt{clan}[B]$ or litter in a $\texttt{litters}[B]$ with $B \ll A$ such that the order on the modified support does not already contain such a support for the item appearing before the item: use the method described above for merging extended supports to merge the supports, placing the modified support last in the order on the disjoint union (the modified support may not be strong, but the merger process repairs this).

We might need to repeat this process, as a B-overextended support just added may contain new atoms in $\texttt{clan}[B_1]$, for which a B_1-extended support will be required if $B_1 \ll A$, but this involves finitely many additional iterations for each support added (an iteration adding many B_1-overextended strong supports, then one adding many B_2-overextended strong supports, etc.), so at most ω steps overall. Further, if we successfully construct Π_A, we can construct an A-overextended strong support for the object in the same way.

5 The extension property: allowable permutations act freely

We begin by proving that the allowable permutations act fairly freely.

Definition: We define an A-*locally small bijection* as an injective map ρ_0 on atoms such that

1. The domain and range of ρ_0 are the same set.
2. The range of the restrictions of ρ_0 and its inverse to each clan with index $\underline{\ll} A$ are included in that clan.
3. The intersection of the domain of ρ_0 with each litter included in a clan with index $\underline{\ll} A$ is small (empty is a special case of small). For each $B \ll A$, the intersection of the domain of ρ with $\texttt{setparents}[B]$ is empty (the domain of ρ may intersect $\texttt{parents}[B]$, but the intersection will only contain regular atoms).

Definition: We define an *exception* of an A-allowable permutation ρ as an atom x belonging to a litter L in a $\texttt{clan}[B]$ with $B \underline{\ll} A$ such that either $\rho(x) \notin \rho(L)^\circ$ or $\rho^{-1}(x) \notin \rho^{-1}(L)^\circ$

Definition: For each A, we define the A-*extension property* as the assertion that every A-locally small bijection ρ_0 can be extended to an A-allowable permutation which has no exceptions other than elements of the domain of ρ_0.

Theorem: For every A, the A-extension property holds (of course on the assumption that Π_B is defined for each $B \ll A$).

Proof of Theorem: Fix a clan index A.

Fix an A-locally small bijection ρ_0. We can suppose without loss of generality that ρ_0 is defined on all elements of $\mathtt{parents}(A)$ and on all members of $\mathtt{parents}(B)$ or $\mathtt{clan}(B)$ where B does not downward extend A.

For each pair of litters L, M which are included in the same clan with index $\lll A$ (we make this choice for all such pairs in all such clans), choose a bijection $\rho_{L,M}$ from $L \setminus \mathtt{dom}(\rho_0)$ to $M \setminus \mathtt{dom}(\rho_0)$. Choose them in such a way that $\rho_{M,N} \circ \rho_{L,M} = \rho_{L,N}$[6].

We claim that for every A, ρ_0, and scheme of maps $\rho_{L,M}$ satisfying the stated conditions, there is a unique A-allowable permutation ρ which extends ρ_0 and extends each $\rho_{L,\rho(L)^\circ}$. This claim we call the strong A-extension property.

Observe that the strong A-extension property holds iff the strong $\{\mathtt{min}(A)\}$-extension property holds, by the properties of the structural correspondence maps. Thus, we can prove that the A-extension property holds using the inductive hypothesis that the strong B-extension property holds for each $B \ll A$ (in effect by induction on the minimum element of the index, with $\emptyset$ occurring at stage λ).

Given an object x and an A-extended support S for x with associated order $<_S$, we describe the computation of a value $\rho^*_S(x)$ (which will turn out to be the value at x of the unique allowable permutation ρ extending ρ_0 and each map $\rho_{L,\rho(L)^\circ}$). The function ρ^*_S is defined by recursion on the order $<_S$.

If x is in a clan $\mathtt{clan}[B]$ with B not downward extending A, define $\rho^*_S(x)$ as $\rho_0(x)$. Otherwise, the litter L containing x is an element of S. We compute the value $\rho^*_S(x)$ as either $\rho_0(x)$ or $\rho_{L,\rho^*_S(L)^\circ}$. Of course, we have to describe how we compute ρ^*_S on the elements of S. If y is an atom in S, it is preceded by the litter L containing y, at which we have already computed ρ^*_S, and we compute $\rho^*_S(y)$ as either $\rho_0(y)$ or $\rho_{L,\rho^*_S(L)^\circ}(y)$.

If L is a litter in S included in $\mathtt{clan}[A]$ with parent p, we compute the image of each atom belonging to the litter L using either ρ_0 or $\rho_{L,(\Pi\lceil\mathtt{litters}[A])^{-1}(\rho_0(p))}$ to obtain $\rho^*_S(L)$. If L is a litter in S included in

[6]This certainly can be done but is no longer required in the argument.

a $\texttt{clan}[B]$ for $B \ll A$ and L does not have set parent, and has atomic parent p, we can compute the image of each atom in L using either ρ_0 or $\rho_{L,(\Pi\lceil\texttt{litters}[B])^{-1}(\rho^*_S(p))}$ to obtain $\rho^*_S(L)$.

Suppose L in S has a set parent. All elements of a B-extended strong support for the set parent of L appear before L in the well-ordering associated with the strong support of x (use the largest possible such support, which is actually overextended), and ρ^*_S has been computed at all elements of this support.

Let ρ' be a B-allowable permutation extending the restriction of ρ_0 to clans with index $\underline{\ll}B$ and the restriction of ρ^*_S as computed so far to $\texttt{parents}[B]$, extending each map $\rho_{U,\rho'(U)^\circ}$ where the index of U downward extends B. Such a map exists by the strong B-extension property, which we have assumed as an inductive hypothesis.

We compute the value of ρ^*_S at L by application of either ρ_0 or

$$\rho_{L,\Pi^{-1}(\Pi_B^{-1}(\rho'(\Pi_B(\Pi(L)))))}$$

to each of its elements.

We have to verify that this last procedure does not depend on the choice of ρ' (the freedom of choice has to do with values at elements of $\texttt{parents}[B]$ which are not in the domain of ρ^*_S as defined so far). If distinct ρ' and ρ'' gave distinct results, they would have to disagree at a first atom x or litter M in the given support of L. If they disagree at an atom x, they agree at the litter N in the support containing it, but then ρ' and ρ'' are computed in the same way at x, using either ρ_0 or $\rho_{N,\rho'(N)^\circ} = \rho_{N,\rho''(N)^\circ}$, so they cannot in fact disagree. Suppose they disagree at a litter M. The maps ρ' and ρ'' agree at each element of a B-support of the set parent of M (if it has one), so they agree at the parent of M (we have already shown that they will agree at the parent if it is a regular atom with index $\underline{\ll}B$, and they clearly will agree at the parent of M if this is in $\texttt{parents}[B]$, as it will also be in the domain of ρ^*_S as computed so far). But then ρ' and ρ'' are computed in the same way at each element of M, using either ρ_0 or $\rho_{M,\rho'(M)^\circ} = \rho_{M,\rho''(M)^\circ}$, so they cannot in fact disagree at M. The same argument shows that ρ' and ρ'' agree with ρ^*_S at each element of the B-support, not only at elements of $\texttt{parents}[B]$ in the B-support, and so will agree on the B-support with the map ρ we are constructing when we succeed in constructing it.

Now we claim that the value of $\rho^*_S(x)$ does not depend on the choice of the support S.

Fix a A-extended support S and consider the $<_S$-first $y \in S$ such that $\rho^*_S(y) \neq \rho^*_T(y)$ for some other A-extended support T for y. It should be evident from the calculations above that y cannot be an atom or a litter included in $\texttt{clan}[A]$ or a litter with regular atomic parent. In the case where y is a litter with set parent included in a $\texttt{clan}[B]$ with $B \ll A$, choose a B-allowable permutation ρ' extending the restriction of ρ_0 to clans with index $\leqq B$, the restrictions of ρ^*_S and ρ^*_T as computed so far to $\texttt{parents}[B]$, which cannot disagree because the values of ρ^*_S computed so far are independent of S by hypothesis, and all maps $\rho_{U.\rho'(U)^\circ}$ where the index of U downward extends B, using the inductive hypothesis that the B-strong extension property holds. This map ρ' meets the required conditions to compute the values of both $\rho^*_S(y)$ and $\rho^*_T(y)$ by the procedure above, so these values must be the same, contrary to hypothesis.

We then define $\rho(x)$ as $\rho^*_S(x)$ for an arbitrary support S for x [and at irregular atoms p define $\rho(p)$ as $\Pi(\rho(\Pi^{-1}(p)))$]; ρ extends to sets by our standard conventions and is forced by the construction to satisfy the conditions to be an A-allowable permutation, and further ρ clearly has no exceptions other than elements of the domain of ρ_0. We remark that if a litter L appearing in an A-extended support set S has set parent $X = \Pi_B(\Pi(L))$, it is mapped to a litter with set parent $\rho'(\Pi_B(\Pi(L)) = \rho'(X)$ as explained above, where ρ' has been shown above to agree with ρ^*_S and so with ρ on a suitably indexed support of X, so $\rho'(X) = \rho(X)$ as required.

6 Counting orbits: the construction succeeds at all stages

Now we need to argue that there are no more than μ elements of each X_α, to ensure that Π_A is defined for each A.

Note that there are μ support sets: there are μ atoms, and there are μ near-litters, and support sets are small sets of atoms and near-litters.

Recall that X_α is defined as the collection of strongly symmetric sets in $\bigcup_{\beta<\alpha} \mathcal{P}^2(\texttt{clan}[\{\alpha, \beta\}])$. In older versions of the argument, this was the collection of strongly symmetric sets in

$$\bigcup_{\mathtt{min}(B)<\alpha} \mathcal{P}^{|B|}(\mathtt{clan}[B]).$$

It would suffice to show that each $\mathcal{P}^{n+1}(\mathtt{clan}[A])$, with $n < |A|$, has μ strongly symmetric elements. Each such set has at least μ such elements: consider iterated singletons of atoms.

We introduce the concept of an *A-coding function*. Let x be an object with an A-strong support S with support order $<_S$. We claim that there is a function $\chi^A_{x,S}$ such that for each A-allowable permutation ρ,

$$\chi^A_{x,S}(\rho(<_S)) = \rho(x).$$

Observe that if $\rho(<_S) = \rho'(<_S)$, it follows that $\rho(x) = \rho'(x)$ because S is an A-support for x ($\rho^{-1} \circ \rho'$ fixes $<_S$, so it fixes each element of S, so it fixes x). The coding function $\chi^A_{x,S}$ has domain an orbit in the A-support orders under the A-allowable permutations, and its range is an orbit in the A-allowable permutations, and every orbit in the A-allowable permutations is the range of such a function (though an orbit will be produced as the range of many coding functions).

Definition: An A-coding function is a function χ whose domain is the orbit in the A-allowable permutations of an A-strong support order $<_S$ and which satisfies the equation $\chi(\rho(<_S)) = \rho(\chi(<_S))$ for each A-allowable permutation ρ. The notation $\chi^A_{x,S}$ is used for this coding function, where $x = \chi(<_S)$. We make a further technical restriction: a coding function whose range is included in the power set of a clan will have as elements of its domain support orders whose domains include only atoms in that clan and near-litters included in that clan.

We aim to show that for each iterated power set of a clan $\mathcal{P}^{n+1}(\mathtt{clan}[A])$, $n < |A|$, there are $< \mu$ A_n-coding functions with range included in the iterated power set, which implies that the iterated power set is of size no more than μ (since the range of each coding function is of size no more than μ) and therefore exactly μ. We will then further show that there are $< \mu$ A_{n+1}-coding functions whose ranges are included in the same set, to support the induction.

We first discuss the case $n = 0$ using the considerations above.

Analysis of orbits and coding functions in power sets of clans : We claim that the orbits in $\texttt{clan}[A]$ under B-allowable permutations ($A \lll B$) fixing a B-strong support S are the singletons $\{x\}$ for $x \in S$, near-litters $L \setminus S$ for $L \in S$, and the additional orbit $\texttt{clan}[A] \setminus (S \cup \bigcup S)$. Obviously if an atom $x \in \texttt{clan}[A]$ belongs to S, $\{x\}$ is an orbit. If x and y are atoms in $L \setminus S$ where L is a near-litter in S included in $\texttt{clan}[A]$, or if x and y are atoms in $\texttt{clan}[A]$ not belonging to any litter in S, (with x and y in either case not anomalies of near-litters in S) then the map exchanging x and y and fixing all elements of $\texttt{parents}[B]$ and all atoms in S and all anomalies of near-litter elements of S is a B-locally small bijection and can readily be shown to be extendible to a B-allowable permutation fixing all elements of S by the extension property: this verifies the identities of the orbits. To verify that near-litters in S are fixed by this permutation, suppose that a $<_S$-first near-litter M is not fixed: if M does not have set parent, its parent is clearly fixed; if M has set parent, the set parent is fixed because all elements of a suitably indexed support for the set parent are fixed; then observe that the near-litter M itself must be fixed because M is fixed iff the litter M° is fixed (because we fix the anomalies of M) and the permutation has no exceptions which it moves, other than possibly x and y, and if either x and y belonged to M, both would belong to M (and in fact to M°) and they would be exchanged by the map within M and so would not be exceptions. It follows from this that the subsets of a clan in the FM interpretation will be exactly the sets X for which there is a support S such that X is a union of orbits in the permutations fixing all elements of S, which can be described generally as the sets which have small symmetric difference from small or co-small unions of litters. Note that this shows that every subset of $\texttt{clan}[A]$ has a support consisting entirely of atoms and litters with index A: a set with small symmetric difference from a small or co-small union of litters clearly has a support consisting entirely of atoms in the clan and litters included in the clan. The stronger condition holds that for any support S for a subset X of $\texttt{clan}[A]$, $S \cap (\texttt{clan}[A] \cup \texttt{nearlitters}[A])$ is also a support for X.

Choose a B-strong support S for an element of $\mathcal{P}(\texttt{clan}[A])$, with order $<_S$. Let $<_{S'}$ be the restriction of $<_S$ to objects with index A (the domain S' of this order is in fact a support of the set by considerations in the previous paragraph). A B-symmetric set will be determined by a sequence of bits one longer than $<_{S'}$, with each bit other than the last telling us whether to include the orbit associated with the corresponding support element in $<_{S'}$ in the set produced (the orbit associated with an atom x being its singleton and the orbit associated with a near-litter L being $L \setminus S$, as above) and the last one telling us whether to include the orbit $\texttt{clan}[A] \setminus (S \cup \bigcup S)$ in the set produced. Clearly each such function is a coding function, and there are $< \kappa$ of them for each support S. In the case $B = A$, the order $<_{S'}$ is in fact an A-extended strong support, and (since A-allowable permutations act completely freely on litters in $\texttt{litters}[A]$ and quite freely on small sets of atoms in $\texttt{clan}[A]$, by the extension property) there are $< \kappa$ orbits in such support orders: the orbit of an order $<_{S'}$ is entirely determined by its length and whether each item in it is an atom or near-litter, so there are $< \kappa$ such orbits, and $< \kappa$ A-coding functions. We will show below in the main argument why there are $< \mu$ orbits in the support orders $<_S$ in the case $A \lll B$ (e.g., $B = A_1$), and so $< \mu$ B-coding functions with range included in $\mathcal{P}(\texttt{clan}[A])$.

Note that these results show that we are free to restrict our attention to coding functions for subsets of clans whose domain elements are support orders whose elements all have the same index as the clan, since every support of a subset of a clan includes a support of this kind.

Definition (specifications of support orders, overspecified supports): We next discuss a formal way of specifying the orbits under A- or A_1-allowable permutations of A-strong support orders. An A-strong support order $<_S$ can be described in terms of certain data: we intersperse the informal description of this data with the description of the components of the mathematical object representing this data, a *specification* ξ_S of an A-strong support S, which is a function with domain the order type of S.

We define $S(\alpha)$ as the object in position α in the order $<_S$.

1. whether the item in each position is an atom or a near-litter, and the index of the set $\texttt{clan}[B]$ or $\texttt{nearlitters}[B]$ to which it belongs.

   The first projection of $\xi_S(\alpha)$ will be 0 if $S(\alpha)$ is an atom and 1 if

$S(\alpha)$ is a near-litter. The second projection of $\xi_S(\alpha)$ will be the index of $S(\alpha)$.

2. if the item is an atom, the ordinal index of the position at which a near-litter containing it is present (this is unique because distinct near-litters in a support set are disjoint).

   If $\pi_1(\xi_S(\alpha)) = 0$, then the third projection β of $\xi_S(\alpha)$ satisfies $S(\alpha) \in S(\beta)$ and $\beta < \alpha$. The fourth projection of $\xi_S(\alpha)$ will be 0.

3. if the item is a near-litter without set parent in a $\texttt{clan}[B]$ with $B \ll A$, the ordinal position at which its atomic parent is present.

   If $S(\alpha)$ is a near-litter without set parent, the third projection of $\xi_S(\alpha)$ is 0. If the index of $S(\alpha)$ is A, the fourth projection of $\xi_S(\alpha)$ is 0. Otherwise, the fourth projection of $\xi_S(\alpha)$ is $\beta < \alpha$ such that $S(\beta)$ is the regular atomic parent of $S(\alpha)$.

4. if the item is a near-litter included in $\texttt{clan}[B]$ with set parent, a B-coding function and the specification of the indices of items in a suborder of the segment of $<_S$ before the item such that applying the coding function to the sublist yields the set parent of the item. If one is only interested in obtaining an orbit under A-allowable permutations, one may restrict this to the case $B \ll A$.

   If $S(\alpha)$ is a near-litter included in $\texttt{clan}[B]$ with set parent X, the third projection of $\xi_S(\alpha)$ is 1, and the fourth projection of $S(\alpha)$ is of the form (χ, E) where χ is a B-coding function, E is a set of ordinals $< \alpha$, and $\chi(<_S \lceil (S``E)) = S(\alpha)$. $S``E$ abbreviates $\{S(\beta) : \beta \in E\}$. Moreover, E is the largest set of ordinals for which this can be done.

   If one is only interested in obtaining an A-orbit, one treats the case $B = A$ differently: if $S(\alpha)$ is a near-litter in $\texttt{clan}[A]$, set the third and fourth projections of $\xi_S(\alpha)$ to zero.

We say that an A-support is an A-*overspecified support* if it has a specification satisfying the last condition in its unrestricted form[7]. An A-overextended support is A-overspecified but the converse is not necessarily true (it may have near-litters which are not litters in its domain).

[7]This is another new notion, not to be confused with "overextended", though it is related.

A-specifications correspond exactly to A-orbits of support orders : That $\rho(<_S)$ will have the same specification as $<_S$ if ρ is an A- or A_1-allowable permutation (in the latter case supposing the support to be overspecified) should be evident.

Suppose that $<_S$ and $<_T$ have the same specification [and if the aim is to construct an A_1-allowable permutation, that the specification witnesses that S and T are A-overspecified]. Construct a locally small bijection ρ_0 with the following properties:

1. The map ρ_0 sends an atom at position α in the order $<_S$ to the atom at the same position α in $<_T$.
2. The map ρ_0 sends each element of `parents`$[A]$ or `clan`$[A_1]$ (the former if building an A-allowable permutation, the latter if building an A_1-allowable permutation) which is the parent of a near-litter at position α in $<_S$ to the parent of the near-litter at the same position α in $<_T$.
3. Each anomaly of a near-litter in S and each anomaly of a near-litter in T is in the domain of ρ_0.
4. Any element of a near-litter at position α in $<_S$ which is in the domain of ρ_0 is mapped by ρ_0 to an element of the corresponding near-litter at position α in $<_T$. Any non-element of a near-litter at position α in $<_S$ which is in the domain of ρ_0 is mapped by ρ_0 to a non-element of the corresponding near-litter at position α in $<_T$.

To enforce the condition that the domain and range of a locally small bijection are the same, it may be necessary to add a small collection of additional atoms to the domain of ρ_0 over and above those explicitly required: no more than countably many such additional atoms are required for each of the atoms explicitly required by the conditions, to fill out orbits: images and preimages of each additional atom need to be chosen subject to the last condition. Construct an extension of ρ_0 to an A- or A_1-allowable permutation ρ as appropriate with no exceptions outside the domain of ρ_0: this map will send $<_S$ to $<_T$. The argument for this is of a kind we see several times: if there is an element of the domain of $<_S$ not mapped to the corresponding element of $<_T$ there must be a first one. It must be a near-litter L. Its parent must be mapped to the parent of the corresponding item in $<_T$, if it is a regular atom or if it is a member of `parents`$[A]$. We show that this is also true if L has set parent X. The set parent X will have a strong support $U \subseteq \{x : x <_S L\}$ with $<_U \subseteq <_S$. The set parent X' of the litter L' at the corresponding position in $<_T$ has a corresponding support $<_{U'}$ lying in $<_T$ just as $<_U$ lies in $<_S$. Because L is supposed to be the first bad element, we have $\rho(<_U) = <_{U'}$. But then, because $<_S$ and $<_T$ have the same specification, there is a coding function χ readable from the common specification of $<_S$ and $<_T$ such that $X = \chi(<_U)$ and $X' = \chi(<_{U'})$, so $X' = \chi(<_{U'}) = \chi(\rho(<_U)) = \rho(\chi(<_U)) = \rho(X)$. Finally, L must actually be mapped to the corresponding near-litter L' in $<_T$ exactly because of the tight control we have over exceptions: if $\rho(L) \neq L'$, then, because the anomalies of L are sent to elements or non-elements of L' as appropriate, it must be the case that there is an exception of ρ in L° or $(L')^\circ$, an $x \in L$ with $\rho(x) \notin L'$, or an $x \notin L$ with $\rho(x) \in L'$. But such an x must be in the domain of ρ_0, because all exceptions of ρ must be in this domain, and no element of ρ_0 has this behavior, by the last listed condition on ρ_0.

It can further be noted that there is nothing special about A_1: if an A_1-allowable permutation can be constructed (i.e., if the specification is overextended), we can construct a C-allowable permutation which sends $<_S$ to $<_T$ for any C with $A_1 \underline{\ll} C$, in exactly the same way.

Observations on orbits of overspecified support orders: It is important to note that as soon as we have Π_A defined, we can see what the orbits of the A-overspecified support orders are under C-allowable permutations for every C with $A \ll C$, when and if we succeed in defining Π_C, though we may be very far from being able to define Π_C yet: they are precisely the same as the A_1-orbits. It follows that as soon as we know what Π_{A_n} is we can specify C-coding functions for a strongly symmetric subset of $\mathcal{P}^{n+1}(\texttt{clan}[A]$ for every C with $A_n \ll C$ whether we have defined Π_C yet or not, because in fact they will be the same functions, due to the fact noted above that the orbits in A_n-overspecified support orders in C-allowable permutations are the same for all such C.

Theorem: The collection of strongly symmetric elements of each $\mathcal{P}^{n+1}(\texttt{clan}[A])$, where $n < |A|$, has cardinality μ, and so each X_α has cardinality μ and

Π_A can be defined for every clan index A.

Proof of the theorem: We consider the counting of specifications. Specifications are small sequences of items built from clan indices, small ordinals, and coding functions. Specifications for a given A-support will involve B-coding functions for set parents of elements of $\texttt{litters}[B]$ for $B \lll A$, that is, elements of $\mathcal{P}^m(\texttt{clan}[C])$'s for which $C_m \lll A$ – and if we are in the case where $C_m = A$ is allowed, elements of the domains of A-coding functions involved in the specification of the given A-support will have order type less than the order type of the given A-support.

Thus, we can deduce that there are $< \mu$ specifications for A-supports, with corresponding information about the number of orbits in the support orders, if we know that there are $< \mu$ C_m-coding functions with range included in the strongly symmetric elements of each $\mathcal{P}^m(\texttt{clan}[C])$ with $C_m \ll A$.

Further, we can deduce that there are $< \mu$ specifications for A-supports which are overspecified, of length a fixed $\alpha < \kappa$, with corresponding information about the number of orbits in the support orders, if we know that there are $< \mu$ C_{m+1}-coding functions with range included in the strongly symmetric elements of each $\mathcal{P}^m(\texttt{clan}[C])$ with $C_m \ll A$, and further we know that there are $< \mu$ A_1-coding functions with range included in the strongly symmetric elements of each $\mathcal{P}^m(\texttt{clan}[C])$ with $C_m = A$ which have A-overspecified strong support S with the order type of $<_S$ less than α.

Now we argue that the there are $< \mu$ A_n-coding functions with range included in the strongly symmetric elements of $\mathcal{P}^{n+1}(\texttt{clan}[A])$ (so the collection of strongly symmetric elements has cardinality μ), and then that there are $< \mu$ such C-coding functions for any C with $A_n \ll C$, in two passes with different qualifications in each pass. In the first pass, we assume just that there are $< \mu$ C_{m+1}-coding functions with range included in the strongly symmetric elements of each $\mathcal{P}^{m+1}(\texttt{clan}[C])$ for which $C_m \ll A_n$ [note that this implies that we successfully define Π_C where $C \ll A_n$], and show that there are $< \mu$ A_n-coding functions with range included in the strongly symmetric sets in $\mathcal{P}^{n+1}(\texttt{clan}[A])$. This is enough to show that Π_A can be defined.

In the second pass, we fix an ordinal α and assume, in addition to the hypothesis above, that there are $< \mu$ A_{n+1}-coding functions with range included in the strongly symmetric elements of $\mathcal{P}^{n+1}(\texttt{clan}[A])$ which have A_n-overextended support of length $< \alpha$, and show that there are $< \mu$ A_{n+1}-coding functions with range included in the strongly symmetric elements of $\mathcal{P}^{n+1}(\texttt{clan}[A])$ which have A_n-overspecified support of length $\leq \alpha$. This result shows (when carried out for all α) that there are $< \mu$ A_{n+1}-coding functions with range included in the strongly symmetric elements of $\mathcal{P}^{n+1}(\texttt{clan}[A])$ (the stronger result being needed for the induction to continue, as we will see below).

If $n = 0$, we are counting coding functions for subsets of a clan. These are specified as discussed above by a sequence of $< \kappa$ bits and an orbit in A_n-supports (overspecified of length $\leq \alpha$ in the second pass). We have inductive hypotheses sufficient to deduce that there are $< \mu$ orbits in the A_n-supports in each pass of the kind that that pass requires, and so that there are $< \mu$ A_n- or A_{n+1}-coding functions whose ranges cover as much of $\mathcal{P}^{n+1}(\texttt{clan}[A])$ as is being considered. So it suffices to assume $n > 0$ for the rest of the discussion.

Fix an element x of $\mathcal{P}^{n+1}(\texttt{clan}[A])$ $(n > 0)$ and an A_n-extended support $<_S$ of x [in the second pass, an A_n-overextended support of length $\leq \alpha$].

For each $y \in x$, choose a coding function $\chi^{A_n}_{y,T_0}$ from the $< \mu$ [by ind hyp] A_n-coding functions with range included in $\mathcal{P}^n(\texttt{clan}[A])$, T_0 being an A_{n-1}-overextended support. Merge the orders $<_S$ and $<_{T_0}$ ($<_S$ being placed first): an end extension of $<_S$ will be obtained; then delete all elements of the domain of the resulting order which do not have index downward extending A_{n-1} to obtain an overextended support T for y with support order $<_T$ which is an end extension of the restriction of $<_S$ to items with index downward extending A_{n-1}. It is useful to note at this point that if one takes a C_1-extended support set and drops from it all items which do not have index downward extending C, one obtains a C-*over*extended support set: T is the merger of the A_{n-1}-overextended T_0 and a subsupport of S which is A_{n-1}-overextended for the reason stated, which verifies that it is overextended.

Notice that because $<_T$ is overextended and so overspecified, it is sufficient to specify an A_n-orbit. Let K be the set of all coding functions $\chi^{A_n}_{y,T}$ chosen in this way. For $<_U$ with the same specification as $<_S$ (in the sense appropriate to the pass), define $\chi^*_x(<_U)$ as the set of all $\chi(<_V)$ where $<_V$ end extends the restriction of $<_U$ to clans and litter partitions with index downward extending A_{n-1} [and of course has the right specification to be in the domain of χ] and $\chi \in K$. First, χ^*_x is clearly a coding function. Next, $\chi^*_x(<_S)$ obviously includes x as a subset by construction. We show that in fact every element of $\chi^*_x(<_S)$ belongs to x. If $y' \in \chi^*_x(<_S)$ then $y' = \chi^{A_n}_{y,T}(<_{T'})$ for some $y \in x$. Of course $y = \chi^{A_n}_{y,T}(<_T)$ The orders $<_T$ and $<_{T'}$ are overspecified and so share a specification sufficiently fine that there is an A_n-allowable ρ sending $<_T$ to $<_{T'}$ (we described above how to arrange this using the extension property) and fixing each element of $<_S$ (which can be achieved by a further extension of the locally small bijection used). The map ρ will fix x and will send y to y', establishing that $y' \in x$, completing the argument that $\chi^*_x(<_S) = x$.

Now each of the functions χ^*_x obtained in this way is determined by a specification of an orbit in A_n-supports and a collection of coding functions (the original $\chi^{A_n}_{y,T_0}$'s) taken from a supply of coding functions known to be of size $< \mu$) [and the argument shows that any coding function with the intended codomain can be determined by such data]; we have the inductive hypotheses in each pass to support the assertion that there are $< \mu$ specifications for orbits of the kind appropriate to that pass; there are $< \mu$ such sets of coding functions because μ is strong limit. Thus we have shown that there are $< \mu$ coding functions (suitably indexed in each pass) whose ranges are included in $\mathcal{P}^{n+1}(\texttt{clan}[A])$. Note that in the second pass, we have successfully specified an A_{n+1}-coding function, because the A_n-support we are using is overspecified.

We reiterate that something sneaky is going on here: the argument above shows first that there are $< \mu$ coding functions (*not* using A_n-overspecified supports, because we have not defined Π_{A_n} yet, so just A_n-coding functions), which cover all of $\mathcal{P}^{n+1}(\texttt{clan}[A])$, then returns, having the definition of Π_{A_n}, to the finer-grained argument that we have $< \mu$ A_{n+1}-coding functions (equivalently, C-coding functions for each C with $A_n \ll C$) as well, which of course we need for the induction.

This completes the argument that Π_A can be defined for every clan index A, so the entire construction succeeds in building something.

7 Coincidence of notions of symmetry and the formula for ranges of set parent maps

We can now consider the FM interpretation, since we have defined the full notion of ($\emptyset$-)allowable permutation. We prove the perhaps surprising result that the collection of strongly symmetric elements of $\mathcal{P}^{n+1}(\texttt{clan}[A])$ $(n < |A|)$ is exactly the set of hereditarily symmetric elements of $\mathcal{P}^{n+1}(\texttt{clan}[A])$. It is evident that a strongly symmetric set is hereditarily symmetric: the converse is far from obvious. We prove the result by induction on n.

We address the case $n = 0$. By considerations discussed above, the strongly symmetric subsets of a clan are exactly the hereditarily symmetric subsets of a clan: in both cases we obtain exactly the subsets of the clan which have small symmetric difference from a small or co-small union of litters.

Let X be an element of $\mathcal{P}^{n+1}(\texttt{clan}[A])$, where $0 < n < |A|$ (and assume that the result we are proving has already been shown to hold for smaller values of n). Suppose that X has extended support S and is hereditarily symmetric. We claim that X has the restriction S' of S to clans and litter partitions with index $\underline{\ll} A_n$ as an A_n-extended strong support. By inductive hypothesis, all elements of X are strongly symmetric, so X is thus seen to be strongly symmetric, once the claim about S' is verified.

Let ρ be an A_n-allowable permutation which fixes each element of S'.

Let y be an element of x. By inductive hypothesis, we may suppose that y, a hereditarily symmetric element of $\mathcal{P}^n(\texttt{clan}[A])$, is strongly symmetric and so has an A_{n-1}-extended support T. We merge S and T into a single support with an order $<_{S\cup T}$. We define a locally small bijection ρ_0' which agrees with ρ on orbits in ρ of atoms in T, on atoms in S', and fixes each atom in $S \setminus S'$ and each parent of a litter in $S \setminus S'$ which belongs to $\texttt{parents}[\emptyset]$. It also needs to agree with ρ on orbits in ρ of exceptions of ρ which lie in clans which meet T or elements of T. The map defined in this way is a local bijection and sends elements of each litter L in $S' \cup T$ to elements of $\rho(L)$ and non-elements of each litter L in $S' \cup T$ to non-elements of $\rho(L)$. Additional atoms added to the domain to fill out orbits in ρ are all in clans with index $\underline{\ll} A_{n-1}$.

We extend ρ_0' to an allowable permutation ρ' which we claim fixes x and satisfies $\rho(y) = \rho'(y)$, establishing that $\rho(y) \in x$ (and the same argument applied to ρ^{-1} demonstrates that ρ fixes x). Thus S' is an A_n-extended support for x, and x is strongly symmetric as desired.

What needs to be shown to establish that claim is that the extension ρ' behaves correctly at near-litters in the supports. A litter L in T must be sent to $\rho(L)$ by ρ': if there were a litter for which this failed, there would be a $<_{S\cup T}$-first one. Its atomic parent would be sent to the same value by ρ and by ρ' because it either has regular atomic parent satisfying the same condition or all elements of a suitably indexed support for its set parent satisfy the same condition. All exceptions of ρ' or ρ which lie in clans which meet T or elements of T are in the domain of ρ'_0 and so are mapped to the same images by ρ and ρ', or are in S' and fixed by both maps, so the litter L itself would be mapped to the right near-litter $\rho(L)$ by ρ', contrary to hypothesis. Litters in S need to be shown to be fixed. This is similar: a $<_{S\cup T}$-first litter in S not fixed would have all elements of a support for its parent fixed (earlier elements of S) so its parent would be fixed, so it could only be moved if there were an exception of ρ' mapped into or out of the litter, and no litter in S can contain an exception of ρ' [at least, not one that is moved by ρ': such an exception must belong to a clan which meets T or an element of T]: for litters in S' this follows from the fact that ρ fixes these, and for litters in $S \setminus S'$ it follows because such litters cannot contain an atom in a clan meeting T or any element of T. A litter in $S \setminus S'$ with parent in $\texttt{parents}[\emptyset]$ is unproblematically fixed.

From this we are able to draw a rather shockingly circular conclusion. Define $\mathcal{P}_*(X)$ as the collection of hereditarily symmetric elements of X (for X hereditarily symmetric). This is the power set operation of the FM interpretation. We can now draw the conclusion that

$$\texttt{rng}(\Pi_A) = \bigcup_{\beta<\texttt{min}(A)} \mathcal{P}^2_*(\texttt{clan}[A \cup \{\beta\}])$$

or, if we take the older approach

$$\texttt{rng}(\Pi_A) = \bigcup_{B \ll A} \mathcal{P}^{|B|-|A|+1}_*(\texttt{clan}[B])$$

I have made the serious error in earlier presentations of starting out by stating this formula, which raises far too many questions. It is better to let it sneak up on you.

8 Facts about cardinals in the FM interpretation

Define $|X|_*$ as the cardinality of the (hereditarily symmetric) set X in the sense of the FM interpretation. We will use $\leq$ without comment to represent the order on cardinalities in the FM interpretation; this should be clear from context. In the absence of choice, use the method of [10] to define cardinals.

Definition: For any near-litter N, define $[N]$, the *local cardinal* of N, as the set of all near-litters with small symmetric difference from N.

Lemma: For any clan index A, $|\mathcal{P}(\texttt{parents}[A])|_* \leq |\mathcal{P}^2(\texttt{clan}[A])|_*$.

Proof of Lemma: The map F sending each p in $\texttt{parents}[A]$ to $[\Pi^{-1}(p)] \in \mathcal{P}^2(\texttt{clan}[A])$ is invariant under allowable permutations and an injection. Further, $(X \subseteq \texttt{parents}[A] \mapsto \bigcup(F``X))$ is also invariant, and also an injection, because distinct local cardinals are disjoint sets. The latter injective map witnesses the desired cardinal inequality.

Definition: We define $\tau(A)$, for each nonempty clan index A, as $|\mathcal{P}_*^2(\texttt{clan}[A])|_*$.

Lemma: For each clan index A with $|A| > 1$, $2^{\tau(A)} = \tau(A_1)$ (the exponential being computed in terms of the FM interpretation).

Proof of Lemma: $2^{\tau(A)}$ is by definition $|\mathcal{P}_*^3(\texttt{clan}[A])|_*$. By the inequality above, $|\mathcal{P}_*^3(\texttt{clan}[A])|_* \geq |\mathcal{P}_*^2(\texttt{parents}[A])|_*$. Because $\texttt{clan}[A_1]$ is included in $\texttt{parents}[A]$, $|\mathcal{P}_*^2(\texttt{parents}[A])|_* \geq |\mathcal{P}_*^2(\texttt{clan}[A_1])|_* = \tau(A_1)$. Now by the inequality $\tau(A_1) = |\mathcal{P}_*^2(\texttt{clan}[A_1])|_* \geq |\mathcal{P}_*(\texttt{parents}[A_1])|_*$. On either formula for the range of the set parent map, $\texttt{parents}[A_1]$ contains a set the same size (via the invariant map Π_{A_1}) as $\mathcal{P}_*^2(\texttt{clan}[A])$, so $|\mathcal{P}_*(\texttt{parents}[A_1])|_* \geq |\mathcal{P}_*^3(\texttt{clan}[A])|_* = 2^{\tau(A)}$.

9 TST, TST$_n$, and NF defined; natural models of TST$_n$ in the FM interpretation

The simple typed theory of sets TST is the first-order theory with equality and membership, with sorts (traditionally called "types") indexed by the natural numbers, atomic formulas $x = y$ being well-formed iff the sort of x

and the sort of y are the same, and $x \in y$ being well-formed iff the index of the sort of y is the successor of the index of the sort of x. The axioms of TST are extensionality (objects x and y of a sort with positive index are equal iff they have the same elements) and comprehension ($\{x|\phi\}$ exists and is of sort with index the successor of the index of the sort of x for each formula ϕ of the language of TST). TST^n is the subtheory of TST using only the n sorts with index less than n.

A *natural model* of TST (or of TST_n) is a model of TST in which sort 0 is implemented as a set X, each sort i used is implemented as $\mathcal{P}^i(X)$, and the equality and membership relations between the sorts are restrictions of the equality and membership relations of the ambient theory. Of course, if we work inside the FM interpretation, we replace $\mathcal{P}^i(X)$ with $\mathcal{P}^i_*(X)$. It should be clear that the theory of a natural model of TST_n is completely determined by n and the cardinality of X, both in the ground interpretation of ZFA and in the FM interpretation.

Quine's "New Foundations" (NF), so-called after the title of his paper [8], is the first-order theory with equality and membership with the axiom of extensionality (sets with the same elements are the same) and the axiom scheme of stratified[8] comprehension ($\{x \mid \phi\}$ exists for each formula ϕ which can be obtained from a comprehension axiom of TST by dropping distinctions of sort between variables without introducing identifications between variables). Notice that the impossible instance $\{x \mid x \not\in x\}$ of the unsorted comprehension scheme, for example, cannot be obtained from a comprehension axiom of TST in this way. Details of mathematical reasoning in NF and related theories are interesting but no part of our mission here: the reader may look at [9], [1], or [4] for more information about this subject.

For each formula ϕ of the language of TST, define ϕ^+ as the formula obtained by replacing each variable x in ϕ with x^+, where $(x \mapsto x^+)$ is a bijection from the set of all variables to the set of all variables of sort with positive index, and x^+ has sort with index the successor of the index of the sort of x, for all variables x. Specker showed in his 1962 paper [13] that NF is equiconsistent with TST plus the Ambiguity Scheme, the collection of assertions $\phi \leftrightarrow \phi^+$ for each sentence ϕ.

[8]A formula of the language of untyped set theory which can be obtained from a formula of TST by dropping type distinctions between variables (without introducing identifications between distinct variables) is called a stratified formula.

10 NF is consistent

We now observe that the natural model of TST_{n+2} in the FM interpretation whose base type is $\texttt{clan}[A]$ and whose top type is $\mathcal{P}_*^{n+1}(\texttt{clan}[A])$ $(n < |A|)$ is isomorphic to the natural model of TST_{n+2} in the FM interpretation whose base type is $\texttt{clan}[A \cup \{\alpha\}]$ and whose top type is $\mathcal{P}_*^{n+1}(\texttt{clan}[A \cup \{\alpha\}])$ by an application of the map σ_α (which is not a map in the FM interpretation; the isomorphism only exists in the ground interpretation), where α dominates A. This is because each of the types consists exactly of the strongly symmetric sets belonging to the identically indexed iterated power set of the clan in the ground interpretation, and the strongly symmetric sets are accurately copied by the structural correspondence maps.

This further implies that the theory of the natural model of TST_{n+2} in the FM interpretation whose base type is $\texttt{clan}[A]$ and whose top type is $\mathcal{P}_*^{n+1}(\texttt{clan}[A])$ is exactly determined by $A \setminus A_{n+1}$, the set of the smallest $n+1$ elements of A. The condition $n < |A \setminus A_{n+1}| = n+1$ holds, and an isomorphism [in the ground interpretation] between the natural model with base type $\texttt{clan}[A \setminus A_{n+1}]$ and top type $\mathcal{P}_*^{n+1}(\texttt{clan}[A \setminus A_{n+1}])$ and the natural model with base type $\texttt{clan}[A]$ and top type $\mathcal{P}_*^{n+1}(\texttt{clan}[A])$ is then obtained by a finite number of applications of the argument of the preceding paragraph. It thus follows that if $A \setminus A_{n+1} = B \setminus B_{n+1}$, the theory of the natural model with base type $\texttt{clan}[A]$ and top type $\mathcal{P}_*^{n+1}(\texttt{clan}[A])$ is the same as the theory of the natural model with base type $\texttt{clan}[B]$ and top type $\mathcal{P}_*^{n+1}(\texttt{clan}[B])$. Further, this means that the theory of a natural model of TST_n with base type of cardinality $\tau(A) = \mathcal{P}_*^2(\texttt{clan}[A])$ is completely determined by $A \setminus A_{n+1}$ (where of course $|A| > n$).

Let Σ be a finite set of formulas of the language of TST. Let n be chosen so that Σ is a finite set of formulas of the language of TST_n. Define a partition of $[\lambda]^{n+1}$ under which each clan index A in $[\lambda]^{n+1}$ is classified using the truth values of the sentences in Σ in the natural models of TST_n with base type of size $\tau(A)$. This partition of $[\lambda]^{n+1}$ into no more than $2^{|\Sigma|}$ compartments has a homogeneous set B of size $n+2$ by Ramsey's theorem. Now observe that natural models of TST_n with base types of cardinality $\tau(B)$ and $\tau(B_1)$ will have the same theory by homogeneity of the partition (their theories are determined by the smallest $n+1$ elements of B and B_1 respectively, and these $n+1$-element sets belong to the same compartment of the partition), and type 1 in a model with type 0 of size $\tau(B)$ is of size $\tau(B_1)$, so the model of TST_{n+1} with base type of cardinality $\tau(B)$ satisfies the instances of the ambiguity scheme $\phi \leftrightarrow \phi^+$ of Specker (ϕ^+ being obtained from ϕ by raising the index of the sort of each variable), for each $\phi \in \Sigma$. This implies that the full ambiguity scheme is consistent with TST by compactness, and so implies the consistency of NF by Specker's results of [13], 1962.

The reader may recognize that this argument is an adaptation of the argument of Jensen for the consistency of NFU (the theory weakening NF by replacing extensionality with extensionality for nonempty sets) in his paper [7] of 1969. We originally suggested the possibility of such an approach in our paper [3] of 1995.

11 Conclusions and questions

The proof that NF is consistent, given above, will go through if $\lambda = \omega$ and $\kappa = \omega_1$.

If a typed assertion is true in all natural models of TST_n in the FM interpretation with base types of a size $\tau(A)$ for n large enough and satisfying enough Ambiguity, then the corresponding stratified assertion will be true in models of NF obtained from the construction. This is a way of investigating what facts might hold in such NF models; actual NF models are obtained by compactness.

Observe that any small set of hereditarily symmetric sets is hereditarily symmetric, with support obtained by taking the unions of supports of its elements all of whose near-litter elements are litters. If one wants any mathematical structure of a known size to be well-orderable, choose κ larger than the size of that structure. The assertion that the reals can be well-ordered or the axiom of Dependent Choices will hold in the FM interpretation for suitable values of κ (the former if $\kappa = \omega_1$; the latter if $\kappa > c$) and so hold in the models of NF obtained from our construction; one can get relative consistency of stronger results of this kind by increasing κ. The axiom of Denumerable Choice which Rosser assumes in [9] holds with $\kappa = \omega_1$. This means that NF has no interesting (stratified) consequences in arithmetic or the theory of any familiar "small" mathematical structure; choosing κ large enough ensures that the structure looks exactly the same in the FM interpretation as in the ground interpretation, and looks the same in the models of NF obtained from the construction.

Relative consistency with NF of forms of the axiom of choice which don't involve a cardinality bound, such as the Prime Ideal Theorem or the assertion that the universe is linearly ordered, cannot be handled by our present methods.

There is a subtle point to be remembered: the FM interpretation contains the same small sets, and so certainly the same countable sets, as the ground interpretation. It is not the case that an actual model of NF will contain even all of its countable subsets; but stratified combinatorial consequences for models of TST_n of existence in the FM interpretation of all small sets of its domain which exist in the ground interpretation also hold in the models of NF obtained.

Using larger values of λ will allow proof of consistency of stronger extensions of NF, by using values of λ with stronger partition properties. The consistency of the Axiom of Cantorian Sets of Henson ([2]) or the Axioms of Small and Large Ordinals of this author (see [4], [11], [5]) can be established in essentially the same way that is reported in the author's paper [5] on strong axioms of infinity in NFU, assuming that λ is a suitable large cardinal and applying stronger partition properties than Ramsey's theorem.

An important point is that the existence of an ω-model of NF can be established. This can be done by brute force if one is willing to take λ a weakly compact cardinal, as one can then apply the argument to theories of models of TST_n expressed with infinitary conjunctions and disjunctions with $< \lambda$ terms, which yields models of NF with no nonstandard elements of λ. One can show the existence of an α-model for any fixed ordinal α, using considerably less consistency strength but more technical subtlety, by emulating Jensen's techniques in the original NFU consistency paper [7].

The existence of an ω-model of NF settles the old question of Maurice Boffa concerning the existence of an ω-model of TNT, the version of TST with sorts indexed by all integers, proposed by Hao Wang in [14]. An ω-model of NF immediately gives an ω-model of TST.

The existence of an ω-model also settles the esoteric question of whether the existence of cardinals with Specker trees of infinite rank is consistent with ordinary set theory. We explain this question and related known results briefly. The Specker tree of a cardinal μ has μ as its top node; each node is a cardinal and the children of a node ν are the preimages of ν under $(\kappa \mapsto 2^\kappa)$. Thomas Forster has shown, by refining an argument of Sierpinski (see [1], p. 48), that all Specker trees are well-founded, even in the absence of Choice, so every Specker tree has an ordinal rank in an obvious sense. Under Choice, the rank of every Specker tree is finite. In NF + Rosser's Axiom of Counting (an axiom originally proposed in [9] which holds in an ω-model) the cardinality of the universe can be proved to have infinite Specker rank. Up until now, it was unclear how one would construct a cardinal of infinite Specker rank in a set theory of the usual kind. If λ is uncountable, $\tau(A)$ for any A with an infinite minimum element has Specker tree of infinite rank in the FM interpretation that we exhibit above. This establishes consistency of existence of cardinals of infinite Specker rank with ZFA; we are confident that standard methods can be used to port this result to ZF.

We note that our paper thus shows the relative consistency of the system of Rosser's book [9], as we have indicated how to choose parameters to get his additional axioms (Denumerable Choice and his Axiom of Counting) to hold. We are happy about this because [9] is a lovely book about logic as the foundation of mathematics, which we commend to the reader, but it has been under a cloud since Specker's disproof of AC in NF in [12].

The results of this paper establish that NF is not very strong. We continue to believe that it is no stronger than TST with the Axiom of Infinity, which is of the same strength as Zermelo set theory with bounded separation. However, our results here do not establish this upper bound on the consistency strength of NF, as our argument in its present form requires the existence of a strong limit cardinal of cofinality ω_1.

This work does not answer the question as to whether NF proves the existence of infinitely many infinite cardinals (discussed in [1], p. 52). A model with only finitely many infinite cardinals would have to be constructed in a totally different way. We conjecture on the basis of our work here that NF probably does prove the existence of infinitely many infinite cardinals, though without knowing what a proof will look like.

A natural general question which arises is, to what extent are *all* models of NF like the ones indirectly shown to exist here? Do any of the features of this construction reflect facts about the universe of NF which we have not yet proved as theorems, or are there quite different models of NF as well?

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