New Foundations is consistent

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Quick description of New Foundations

The simple theory of types (TST) is the first-order theory with equality and membership, with sorts indexed by the natural numbers, with atomic formulas meeting the schematics

$$x^i = y^i, x^i \in y^{i+1},$$

and with axioms of extensionality (objects of positive type with the same elements are equal) and comprehension (for any formula ϕ,

$$\{x^i : \phi\}^{i+1}$$

exists).

This is based on a nice intuitive picture of the universe as containing individuals, sets of individuals, sets of sets of individuals, and so forth. It is usual to add axioms of Infinity and Choice to this theory, but we will not need to do this, for different reasons for each axiom.

Typical Ambiguity

Suppose we provide a bijection $x \mapsto x^+$ between variables and variables of positive type, with x^+ always one type higher than x. Let ϕ^+ be the result of replacing every variable with its image under this map. If ϕ is a formula, ϕ^+ is a formula. If ϕ is a theorem, ϕ^+ is a theorem. If $\{x^i : \phi\}^{i+1}$ is a set defined as an instance of comprehension, $\{x^{i+1} : \phi^+\}^{i+2}$ is a precisely analogously defined object one type higher (here we are thinking of $x \mapsto x^+$ as just incrementing the type index on the variable).

Quine observed in 1937 that this reflection of theorems and defined objects in every type seemed excessive, and one might as well suppose that the types are all the same. The theory he proposed, New Foundations (NF), is the one-sorted first order theory with equality and membership whose axioms are extensionality and "stratified comprehension" ($\{x : \phi\}$ exists for each formula ϕ which can be made a well-formed formula of TST by a suitable assignment of types to its variables.

Relevant things that are known

Specker showed in 1962 that NF is equiconsistent with TST + the Ambiguity Scheme $\phi \leftrightarrow \phi^+$, which is a kind of verification of the intuition behind the proposal.

Specker had showed earlier in 1954, shockingly, that NF disproves the axiom of choice. This cast doubt on the entire project. It also showed that NF proves Infinity, so it is stronger than one expects.

Jensen showed in 1969 that the slight (?) modification of NF restricting extensionality to objects with elements (allowing urelements) produces a theory which is consistent with Choice and Infinity and does not prove Infinity. This is an indication that the obvious oddities of NF (the unusual ways that it foils the paradoxes, the presence of the universe, Frege cardinals, Russell-Whitehead ordinals) are mere distractions: all of these things are present and can be seen to work in NFU.

Tangled webs of cardinals

In 1995, I showed that the existence of certain patterns of cardinals in choice-free ZF (indeed, choice free Mac Lane set theory) implies the consistency of NF. This was done by reverse engineering Jensen's NFU proof.

Let λ be a limit ordinal. It could be ω.

We will call finite subsets of λ *clan indices* throughout this paper, for reasons evident later. If A is a nonempty clan index, we define A_1 as $A \setminus \{\mathtt{min}(A)\}$; for appriopriate A, we define A_0 as A, and A_{n+1} as $(A_n)_1$.

We work in ZF (weaker theories could be used). We use Scott cardinals in the absence of Choice.

We define TST_n as the subtheory of TST mentioning only types $< n$. A natural model of TST_n is one in which the set implementing type $i+1$ is the power set of the set implementing type i and the membership and equality relations of the model are restrictions of the actual membership and equality relations. The theory of a natural model depends only on the cardinality of the set implementing type 0.

A *tangled web of cardinals* is a function τ whose domain is the set of nonempty clan indices and whose range is a set of cardinals with the following properties:

naturality: $2^{\tau(A)} = \tau(A_1)$ when $|A| \geq 2$.

elementarity: For each A with $|A| > n$, the theory of any natural model of TST_n with type 0 implemented as a set of size $\tau(A)$ is determined by $A \setminus A_n$ (the n smallest elements of A).

Theorem (Holmes, 1995): if there is a tangled web, then NF is consistent

Let Σ be a finite set of formulas of the language of TST. Choose an n such that Σ is a set of formulas of the language of TST_n.

We define a partition of $[\lambda]^n$. An $A \in [\lambda]^n$ belongs to a compartment of the partition determined exactly by the truth values of the sentences in Σ in a natural model of TST_n based on a set of cardinality $\tau(A \cup \{\max(A)+1\})$. Notice that this is a partition of $[\lambda]^n$ into no more than $2^{|\Sigma|}$ compartments. There is a homogeneous set B of cardinality $n+2$ for this partition. It is then straightforward to see that the natural models based on a set of size $\tau(B)$ and on its power set of size $\tau(B_1)$ have the same truth values for all formulas in Σ, so the subscheme of the Ambiguity Scheme with $\phi \in \Sigma$

is consistent with TST_{n+1}, so the Ambiguity Scheme is consistent with TST by compactness, so NF is consistent by the theorem of Specker cited above.

First approximation to the FM construction

We have constructed a model of ZFA in which there is a tangled web of cardinals, by a Fraenkel-Mostowski construction (weaker theories can be used: the current version requires the existence of a strong limit cardinal with cofinality ω_1 but we believe that the strength of NF will ultimately prove to be the minimum possible, that of TST with infinity). We start in ZFA + Choice with atoms described shortly.

We give the initial toplevel description of the permutations used.

Let κ be a regular uncountable cardinal (it could be ω_1. Let μ be a strong limit cardinal greater than λ or κ with cofinality at least κ. Sets of cardinality $< \kappa$ are called small and all other sets are called large.

The atoms are partitioned into sets, all of size μ, called $\texttt{clan}[A]$ for each clan index A, $\texttt{parents}[A] \setminus \texttt{clan}[A_1]$ for each nonempty clan index A, and $\texttt{parents}[\emptyset]$. Of course

$$\texttt{parents}[A] = (\texttt{parents}[A] \setminus \texttt{clan}[A_1]) \cup \texttt{clan}[A_1]$$

for each nonempty clan index A.

Each clan is partitioned into sets of size κ called litters. Subsets of clans with small symmetric difference from litters are called "near-litters". The set of litters included in $\texttt{clan}[A]$ is called $\texttt{litters}[A]$ and the set of near-litters included in $\texttt{clan}[A]$ is called $\texttt{nearlitters}[A]$. A map Π is provided whose domain is the set of all litters in all clans, and whose restriction to each $\texttt{litters}[A]$ is a bijection from $\texttt{litters}[A]$ to $\texttt{parents}[A]$. If N is a near-litter with small symmetric difference from L, $\Pi(L)$ is termed the parent of N.

For each nonempty clan index A, we provide a bijection Π_A from $\texttt{parents}[A] \setminus \texttt{clan}[A_1]$ to a range to be specified later: we stipulate at this point that all elements of the range are sets, and we call $\Pi_A(\Pi(L))$ the *set parent* of any near-litter with small symmetric difference from L, when this is defined (some litters have $\Pi(L) \in \texttt{clan}[A_1]$: these litters do not have set parent).

Atoms belonging to clans are called *regular atoms*; all other atoms are called *irregular*.

A permutation π of the atoms is extended by convention to sets by the rule $\pi(A) = \pi``A$.

We say that a permutation π of the atoms (always understood as extended above) is an *allowable permutation* iff

1. Clans are fixed by π.

2. If L is a litter, $\pi(L)$ is a near-litter with parent $\pi(\Pi(L))$.

3. Each map Π_A is fixed by π: so if L has set parent p, $\pi(L)$ has set parent $\pi(p)$.

A support set is a small set of regular atoms and near-litters, whose near-litter elements are pairwise disjoint.

An object x has support S iff each allowable permutation π such that $\pi(s) = s$ for each $s \in S$ satsifies $\pi(x) = x$.

An object is *symmetric* iff it has a support. Note that regular atoms trivially have support and irregular atoms have support the singleton of a litter with the atom as a parent. An object is hereditarily symmetric iff it is either an atom or all elements of its transitive closure are symmetric.

Standard results show that the class of hereditarily symmetric sets is a model of ZFA (in which Choice will be false).

Brief remarks about our aims

We find it useful at this point to briefly state intent.

We expect (though one must work to show this) that the allowable permutations act freely enough that the individual litters are κ-amorphous (they have only small and co-small subsets in the FM model).

All small sets of the original ZFA are sets of the FM model (this is valuable as we can vary κ to get local versions of choice for example).

The purpose of the maps Π_A is to force the parent sets to be at least as large as the so far unspecified ranges of the maps Π_A.

Local cardinals and parent sets

For each litter L, define the *local cardinal* $[L]$ of L as the set of near-litters with small symmetric difference from L.

On reasonable assumptions about how allowable permutations act, we expect $[L]$ to be the collection of subsets of the clan including L of the same cardinality as L in the FM world, thus the name.

There is an obvious symmetric (indeed invariant) map sending $\Pi(L)$ to $[L]$ for all L. Since the sets $[L]$ are pairwise disjoint, this means that there is an invariant injection (for each nonempty A) from $\mathcal{P}(\texttt{parents}[A])$ into $\mathcal{P}^2(\texttt{clan}[A])$ (where the local cardinals live), in the FM world. Thus, in the FM world:

$$|\mathcal{P}(\texttt{parents}[A])| \leq |\mathcal{P}^2(\texttt{clan}[A])|$$

Arranging isomorphisms

We choose maps σ_α for each $\alpha < \lambda$ whose domain is the set of atoms in sets $\texttt{clan}[A]$ and $\texttt{parents}[A]$ with A nonempty and dominated by α. We extend σ_α to sets whose transitive closures contain only atoms in its domain in the obvious way.

For each α and nonempty A dominated by α, σ_α sends $\texttt{clan}[A]$ to $\texttt{clan}[A \cup \{\alpha\}]$ and further σ_α sends litters to litters and $\sigma_\alpha(\sqcap(L)) = \sqcap(\sigma_\alpha(L))$.

Further, we intend that

$$\sigma_\alpha(\sqcap_A(\sqcap(L))) = \sqcap_{A\cup\{\alpha\}}(\sqcap(\sigma_\alpha(L))).$$

This requires that the range of Π_A be a set on which σ_α can act, which we will see it is intended to be.

An effect of this is that the issue of defining the Π_A's reduces to defining the maps $\Pi_{\{\alpha\}}$ for $\alpha < \lambda$.

The effect of the maps σ_α is to provide external isomorphisms (these are not maps of the FM interpretation!) between structures based on $\texttt{clan}[A]$ and structures based on $\texttt{clan}[A \cup \{\alpha\}]$ where α dominates A, in general terms.

The intended tangled web

We intend to define Π_A in such a way as to induce the map $\tau(A) = |\mathcal{P}^2(\mathtt{clan}[A])|$ (in the sense of the FM world) to be a tangled web.

Define $B \ll A$ as holding when A, B are distinct clan indices and all elements of $B \setminus A$ are less than all elements of A.

We do this by arranging for the range of each map Π_A to be

$\bigcup_{B \ll A} \mathcal{P}_*^{|B|-|A|+1}(\mathtt{clan}[B])$

with $\mathcal{P}_*$ denoting the power set operation of the FM world. It is not at all obvious that this can be achieved. It does directly imply that Π_A is included in the range of σ_α for α dominating A. It is arranged if we arrange each Π_α to have range $\bigcup_{B \ll \{\alpha\}} \mathcal{P}_*^{|B|}(\mathtt{clan}[B])$, as application of sigma maps induces the rest of the equations to hold.

Naturality

From this inequality and the admittedly improbable description of the maps Π_A we get the naturality condition for the claimed tangled web. Here we use $|\cdot|_*$ to denote cardinals in the FM world.

What we need to show is that $2^{\tau(A)}$ in the sense of the FM interpretation, that is,

$$|\mathcal{P}_*(\mathcal{P}_*^2(\mathtt{clan}[A]))|_*,$$

is equal to $\tau(A_1))$, that is, $|\mathcal{P}_*^2(\mathtt{clan}[A_1])|_*$.

$$|\mathcal{P}_*(\mathcal{P}_*^2(\mathtt{clan}[A]))|_* = |\mathcal{P}_*^3(\mathtt{clan}[A])|_* \geq$$

$$|\mathcal{P}_*^2(\mathtt{parents}[A])|_* \geq |\mathcal{P}_*^2(\mathtt{clan}[A_1])|_*.$$

This uses the cardinal inequality above and the fact that $\mathtt{parents}[A]$ includes $\mathtt{clan}[A_1]$ as a subset.

Now, in the other direction:

$$|\mathcal{P}_*^2(\texttt{clan}[A_1])|_* \geq |\mathcal{P}_*(\texttt{parents}[A_1]|_*$$

$$\geq |\mathcal{P}_*(\mathcal{P}_*^{|A|-|A_1|+1}(\texttt{clan}[A]))|_* = |\mathcal{P}_*^3(\texttt{clan}[A])|_*.$$

This depends on the fact that $\texttt{parents}[A_1]$ contains a set of the same cardinality in the FM interpretation as $\mathcal{P}_*^2(\texttt{clan}[A])$, this fact being witnessed by (part of) the invariant map Π_{A_1}.

Elementarity

The elementarity conditions hold because of the existence of the external isomorphism σ_α between the transitive closures of $\mathcal{P}^{n+1}(\texttt{clan}[A])$ and $\mathcal{P}^{n+1}(\texttt{clan}[A \cup \{\alpha\}])$ for $n < |A|$ and α dominating A. These maps witness the fact that if $A \setminus A_n = B \setminus B_n$ and $|A|, |B| > n$, the transitive closures of $\mathcal{P}_*^{n+1}(\texttt{clan}[A])$ and $\mathcal{P}_*^{n+1}(\texttt{clan}[B])$ will be externally isomorphic and so the natural models of TST_{n+2} with base types $\texttt{clan}[A]$ and $\texttt{clan}[B]$ have the same theory. But this means that the natural models of TST_n with base types $\mathcal{P}_*^2\texttt{clan}[A]$ and $\mathcal{P}_*^2\texttt{clan}[B]$ in the FM interpretation have the same theory, and these models have bases of cardinality $\tau(A)$ and $\tau(B)$ as desired.

How is this actually achieved?

We very briefly outline how this is achieved.

We say that a permutation π of the atoms (always understood as extended above) is an *A-allowable permutation* iff

1. Clans with index $\underline{\ll} A$ are fixed by π.

2. If L is a litter in a clan with index $\underline{\ll} A$, $\pi(L)$ is a near-litter with parent $\pi(\Pi(L))$.

3. Each map Π_B is fixed by π, where $B \ll A$.

Symmetry with respect to A-allowable permutations is defined as above.

We need a detailed analysis of supports to support an analysis of orbits under the allowable permutations. (In the talk, I cannot talk too much about this).

A strong support set is a support set S equipped with a well-ordering $<_S$ under which each atom x in S is preceded in $<_S$ by the near-litter in S which contains x, if there is one, each near-litter in S whose parent is a regular atom in S is preceded by that regular atom in $<_S$, and each near-litter $L \subseteq \texttt{clan}[C]$ belonging to S has the collection of $<_S$-predecessors of L which belong to or are included in clans with index $\underline{\ll} C$ as a (strong) support of the set parent of L. A-strong support sets are defined similarly, with the qualification that near-litters $L \subseteq \texttt{clan}[C]$ with $C \ll A$ satisfy the support condition. To have strong support or A-strong support is defined in the obvious way.

An extended strong support is one in which each regular atom is preceded by a near litter containing it and each near-litter whose parent is a regular atom is preceded by that atom, and in which each near-litter element is actually a litter.

An A-extended strong support is defined analogously. An A-overextended strong support has the support property for litters with set parents strengthened to apply to litters included in clans with index $\underline{\ll} A$. Note that the definition of an A-overextended strong support set depends on Π_A, while the other A-qualified symmetry notions depend only on information about Π_B for $B \ll A$.

Strong symmetry

We define a strongly symmetric set as a set belonging to a $\mathcal{P}^{n+1}(\texttt{clan}[A])$ with $n < |A|$, having an A_n-extended support.

The advantage of this notion is that it depends on knowledge of Π_B only for $B \ll A_n$.

We then actually construct $\Pi_{\{\alpha\}}$ so that it contains the strongly symmetric elements of $\bigcup_{B \ll \{\alpha\}} \mathcal{P}^{|B|}(\texttt{clan}[B])$.

We construct $\Pi_{\{\alpha\}}$ with somewhat more care:

There are some additional details. For each $\alpha < \lambda$, suppose that we have chosen in advance a well-ordering $<_\alpha$ of $\texttt{parents}[\{\alpha\}]$, of order type μ, and that further we define $<_A$ where $|A| \geq 2$ as $\sigma_{\texttt{max}(A)}(<_{A \setminus \{\texttt{max}(A)\}})$.

Suppose that the set $\texttt{clan}[\emptyset] \cup \bigcup_{B \ll \{\alpha\}} \mathcal{P}^{|B|}(\texttt{clan}[B])$ is of size μ [if it is not, the construction fails at α]. Choose a well-ordering $<_2$ of type μ of this set. Work along the order $<_\alpha$ to assign values of $\Pi_{\{\alpha\}}$: for each x in $\texttt{parents}[\{\alpha\}] \setminus \texttt{clan}[\emptyset]$, choose $\Pi_{\{\alpha\}}(x)$ to be the $<_2$-first object which has an $\{\alpha\}$-extended strong support which contains no litter included in $\texttt{clan}[\{\alpha\}]$ with parent not $<_\alpha x$. This process is guaranteed to map onto the intended range because supports are small and the cofinality of μ is at least κ.

This process works as a recursive definition, because in defining $\Pi_{\{\alpha\}}$ we only need information about Π_B with $B \ll \{\alpha\}$, and we know what Π_B is if we know what $\Pi_{\{\texttt{min}(B)\}}$ is, and $\texttt{min}(B) < \alpha$. We have already seen that construction of $\Pi_{\{\alpha\}}$ allows construction of all the other maps.

Outline of lemmas

We first prove (now that we actually have a definition of the structure) that allowable permutations act quite freely [at least as far as the definitions succeed].

We then prove that the size in the background ZFA + Choice of the intended ranges of the Π_A's is no more than μ, so the construction succeeds. This is done by analysis of the orbits of the allowable permutations, exploiting the exact sense in which they act quite freely.

Finally, we prove that the hereditarily symmetric sets in each $\mathcal{P}^{n+1}(\texttt{clan}[A])$ are exactly the strongly symmetric sets in that power set, using the same tools. This verifies the unlikely recipe for the collections of set parents.

Freedom of action of permutations

We claim that the allowable permutations act quite freely on small sets of atoms (and actually on large sets of atoms which are locally small in a suitable sense).

Define a *locally small bijection* as an injective map with domain equal to its range on atoms which are either regular or belong to $\texttt{parents}[\emptyset]$ whose domain has a small intersection with each litter.

Define an *exception* of an allowable permutation π as a regular atom a belonging to a litter L such that either $\pi(A) \not\in \Pi^{-1}(\pi(\Pi(L)))$ or $\pi^{-1}(A) \not\in \Pi^{-1}(\pi^{-1}(\Pi(L)))$ (an exception is one of the small number of atoms in any given litter which is mapped to or from an element of an unexpected litter by π).

We claim that any locally small bijection π_0 can be extended to an allowable permutation π all of whose exceptions are in the domain of π_0. (Notice that a locally small bijection can be defined on all of $\texttt{parents}[\emptyset]$).

This is proved by induction along extended strong supports, and by induction on clan indices: the A-extension property is proved using the assumption that the B-extension property holds for $B \ll A$ (and the $\emptyset$-extension property, proved last, is the full extension property).