

Notes on Cubic Example

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October 27, 2004

Here follow some notes about the “Cardano” solution of the cubic equation and what it has to tell us about applicability of complex numbers.

We would like to solve the general cubic equation

$$ax^3 + bx^2 + cx + d = 0$$

Of course writing it in this way presupposes an accepting attitude toward negative numbers; otherwise it breaks up into MANY cases!

Step 1: We can divide by a and reduce it to the form

$$x^3 + \frac{b}{a}x^2 + \frac{c}{a}x + \frac{d}{a}$$

Suppose this done: we will consider only equations of the form

$$x^3 + ax^2 + bx + c$$

from this point on.

Step 2: This step is analogous to “completing the square” in the familiar solution of the quadratic. We substitute $x = u - \frac{a}{3}$:

$$\left(u - \frac{a}{3}\right)^3 + a\left(u - \frac{a}{3}\right)^2 + b\left(u - \frac{a}{3}\right) + c =$$

$$\left(u^3 - au^2 + \frac{a^2}{3}u - \frac{a^3}{27}\right) + \left(au^2 - \frac{2a^2u}{3} + \frac{a^3}{9}\right) + \left(bu - \frac{ba}{3}\right) + c$$

I use the formula $(A - B)^3 = A^3 - 3A^2B + 3AB^2 + B^3$ to expand the cubic term.

All you need to notice here is that the square terms cancel: we end up with an equation of the form

$$u^3 + pu + q = 0$$

and if we can solve an equation of this form we can solve the original equation.

Step 3: We pull a rabbit out of our magic hat at this point.

In the equation

$$u^3 + pu + q = 0,$$

make the substitution $u = A + B$.

$$\begin{aligned}(A + B)^3 + p(A + B) + q &= \\ A^3 + 3A^2B + 3AB^2 + B^3 + pA + pB + q &= \\ (A^3 + B^3 + q) + (A + B)(3AB + p) &\end{aligned}$$

This last expression will be equal to zero if we can impose the following conditions on A and B :

$$\begin{aligned}A^3 + B^3 &= -q \\ AB &= -\frac{p}{3}\end{aligned}$$

And we can!

Change this to

$$\begin{aligned}A^3 + B^3 &= -q \\ A^3 B^3 &= -\frac{p^3}{27}\end{aligned}$$

by cubing the second equation, and we see that A^3 and B^3 can be taken to be roots of the quadratic equation

$$v^2 + qv - \frac{p^3}{27}$$

because in general the solution of a system of equations

$$r + s = C$$

$$rs = D$$

will be the solutions of the quadratic

$$v^2 - Cv + D = 0$$

because $(v - r)(v - s) = v - (r + s)v + rs$.

So we can solve by solving the equation

$$v^2 + qv - \frac{p^3}{27}$$

to find roots r and s , whose cube roots will be the variables A and B from above, whose sum will be the root of the equation

$$u^3 + pu + q = 0$$

from above.

The reason that this method enters into the history of extensions of the number system is that the method cannot find certain real roots of equations without calculations that involve complex numbers!

Example:

We worked on the equation

$$x^3 - 15x - 4 = 0$$

in class.

This has no square term, so we are already at step 3.

$p = -15$ and $q = -4$: the auxiliary quadratic

$$v^2 + qv - \frac{p^3}{27}$$

becomes

$$v^2 - 4v + 125 = 0$$

I'll complete the square on this:

$$(v^2 - 4v + 4) + (125 - 4) = 0$$

$$(v - 2)^2 = -121$$

$$v - 2 = \pm\sqrt{-121}$$

$$v = 2 \pm 11i$$

Now the numbers A^3 and B^3 must be $\sqrt[3]{2 + 11i}$ and $\sqrt[3]{2 - 11i}$ respectively. The root $x = A + B = \sqrt[3]{2 + 11i} + \sqrt[3]{2 - 11i}$. Piece of cake, right?

I can't for the life of me figure out how Cardano would have known how to compute the cube roots of $2 \pm 11i$. However, I suspect that he did know that one of the real roots was 4 anyway (Math 147 techniques work just fine to solve this particular cubic), so he could use the information that the answer was 4 to solve...

Here's how I verified that a correct answer is 4 (there are two other answers!). I don't know whether Cardano would have approached it in just this way.

If $r + si$ is a cube root of $2 + 11i$, with r and s real numbers, we have

$$r^3 + 3(r^2)si + 3r(si)^2 + (si)^3 = 2 + 11i$$

so we have

$$(r^3 - 3rs^2) + (3r^2s - s^3)i = 2 + 11i$$

from which we have

$$r^3 - 3rs^2 = 2$$

$$3r^2 - s^3 = 11$$

Notice also that $(r - si)^3$ can be seen to be $2 - 11i$: if we change s to $-s$, the signs of the terms adding to 11 change (they contain odd powers of s) and the signs of the terms adding to 2 do not.

This means that $x = \sqrt[3]{2 + 11i} + \sqrt[3]{2 - 11i} = (r + si) + (r - si) = 2r$; since we expect our root to be 4, we expect r to be 2. Setting $r = 2$ allows us to solve for s :

$$3(2^2) - s^3 = 11$$

implies

$$s^3 = 1$$

so $s = 1$ and we have $\sqrt[3]{2 + 11i} = 2 + i$ and $\sqrt[3]{2 - 11i} = 2 - i$.

It is easy to verify that $(2 + i)^3 = 2 + 11i$.

I showed in class using Maple that the other two roots of this cubic are also real numbers (though not quite as simple) but calculating these roots using Cardano's formula involves even more complicated calculations with complex numbers!

The point is that calculations using algebraic rules which Cardano and his contemporaries fully expected to work gave real (and provably correct) results but only with the help of "imaginary" numbers. As I commented in class, this is rather as if the fictional detective Sherlock Holmes had given material aid in solving a real crime :-)

Things for you to think about: Write down a cubic and try eliminating the square term ("completing the cube"). Everyone in this class should be able to do this... Then set up the auxiliary quadratic

$$v^2 + qv - \frac{p^3}{27}$$

for the square-free cubic. Again, this shouldn't be too bad.

Solving the quadratic will give something which you can write down but which will usually be ugly. But the solution to your original cubic will require you to take the cube root of this (possibly complex!) quantity, which will probably be impractical :-)

But you should be able to carry out this process up to the point of taking that cube root. If the cube root is real, you should actually be able to write down (but not simplify!) a solution to your equation.