

# Notes on the Construction of Number Systems

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## 1 Three Positions

There are three positions on the construction of number systems which we have discussed in class. The boundaries between them are not clear; one thing which is clear is that this is a philosophical, not a mathematical question: all three approaches lead to the same mathematics!

We outline the three positions and use the construction of the positive rationals from the (positive) natural numbers as a laboratory to illustrate the approaches.

**Position 1:** There is no need to construct new number systems at all: all talk of the new systems can be reduced to talk about the old systems.

**Position 2:** There are definite “abstract objects” which should be identified as the new number systems.

**Position 3:** There are many possible implementations of the new number systems, and it is useful to use such an implementation, but the choice of which one to use is a matter of convention.

## 2 Position 1

We describe Position 1 in the context of constructing the positive rationals. Something like this approach can be seen in Euclid, where he succeeds in talking about both rational and irrational proportions without actually alluding to any numbers but the natural numbers.

Our basic predicate is  $m : n :: p : q$ , “ $a$  is to  $b$  as  $c$  is to  $d$ ”, which is defined as being true for natural numbers  $a, b, c, d$  if and only if  $ad = bc$ .

In what follows, we will use  $p$  and later letters as rational variables. Letters earlier than  $p$  and subscripted variables  $p_1, p_2, q_1, q_2 \dots$  will be natural number variables. All uses of rational variables will be explained away as involving references to natural numbers only. The rational variable  $p$  will have a special relation to the natural number variables  $p_1, p_2$  in our interpretation.

We begin with simple sentences about rational numbers:

$p = q$  is explained as meaning  $p_1 : p_2 :: q_1 : q_2$ , or  $p_1q_2 = p_2q_1$ .

$p < q$  is explained as meaning  $p_1q_2 < p_2q_1$ .

$p = q + r$  is explained as meaning  $p_1 : p_2 :: q_1r_2 + q_2r_1 : q_2r_2$ .

$p = qr$  is explained as meaning  $p_1 : p_2 :: q_1r_1 : q_2r_2$ .

Anything we want to say about the rationals can be expressed using these basic ideas and standard logical constructions. Some logical constructions need attention.

It might not be clear that any statement involving complex sums and products can be reduced to a logical combination of sentences of the forms above. We observe that for any sentence  $P(p)$  about rationals, the sentence  $P(q + r)$  is equivalent to  $(\exists r.p = q + r \wedge P(p))$ , and similarly the sentence  $P(qr)$  is equivalent to  $(\exists p.p = qr \wedge P(qr))$ . Repeated application of these two observations can be used to arrange for every occurrence of a sum or product to be in a sentence of one of the forms above, so that the translation can be applied.

Quantifiers “for all rationals  $p$ ” and “for some rational  $p$ ” need to be explained. If a sentence  $P(p)$  about a rational  $p$  translates to the statement  $P^*(p_1, p_2)$  about two natural numbers  $p_1, p_2$ , we translate  $(\forall p.P(p))$  as  $(\forall p_1.(\forall p_2.P^*(p_1, p_2)))$  and  $(\exists p.P(p))$  as  $(\exists p_1.(\forall p_2.P^*(p_1, p_2)))$ .

We give a worked-out example. The commutative law of addition for rational numbers is  $(\forall p.\forall q.p + q = q + p)$ .

This converts to  $(\forall p.(\forall q.(\exists r.r = p + q \wedge r = q + p)))$ .

(we converted the complex sentence  $p + q = q + p$  to a form only involving the basic sentences defined above)

In turn, this converts to

$$(\forall p_1.(\forall p_2.(\forall q_1.(\forall q_2.(\exists r_1.(\exists r_2.(r_1 : r_2 :: p_1q_2 + p_2q_1 : p_2q_2) \wedge (r_1 : r_2 :: q_1p_2 + q_2p_1 : q_2p_2))))))))))$$

Here we applied the definition to the basic sentences and expanded the quantifiers.

There are two things to observe here. The last statement is a statement entirely about natural numbers, though it appears to express the same fact as the commutative law of addition for rational numbers. This might provide support to a philosophical view that we don't need to believe that there are such objects as "rational numbers", because we can reduce all talk about them to natural numbers.

The second thing to observe is that of course this statement is a true statement about the natural numbers. If we let  $r_1 = p_1q_2 + p_2q_1$  and  $r_2 = p_2q_2$ , then the first proportion  $r_1 : r_2 :: p_1q_2 + p_2q_1 : p_2q_2$  becomes trivially true (it becomes  $p_1q_2 + p_2q_1 : p_2q_2 :: p_1q_2 + p_2q_1 : p_2q_2$ ) and the second becomes  $p_1q_2 + p_2q_1 : p_2q_2 :: q_1p_2 + q_2p_1 : q_2p_2$ , which is almost as trivial: one needs to apply the commutative laws of addition and multiplication for natural numbers to convert it to the trivial form  $p_1q_2 + p_2q_1 : p_2q_2 :: p_1q_2 + p_2q_1 : p_2q_2$ .

The commutative law for rational numbers is seen to be expressible as a fact about natural numbers, but note that it is not the same fact as commutativity of addition for naturals: it also requires commutativity of multiplication for naturals.

I will insert the statement that positive rationals have multiplicative inverses (another example I did in class) as a further example at this point as I update this file.

### 3 Positions 2 and 3

In Positions 2 and 3, we do construct new objects which serve for us as positive rationals, positive reals, and general real numbers. The difference between Position 2 and Position 3 is that the advocate of Position 2 seeks a construction which gives objects which are *really* the positive rationals (for example). Russell seems to have this attitude at least about his definitions for the (positive) rationals and integers. The advocate of Position 3 recognizes that there are various implementations of (say) the positive rationals, but doesn't recognize any of them as *the* implementation. Position 3 is the most usual attitude nowadays.

These positions are not altogether mutually exclusive. It is clear that talk of rational numbers or integers *can* be reduced to talk of natural numbers alone, whether one has a favorite construction of rationals or integers or not. Talk of reals is reducible to talk of natural numbers and sets of natural numbers, and for most mathematicians, sets of natural numbers are just as

serious a commitment as real numbers.

[I note that the infinite sets of natural numbers correspond precisely to the reals in the interval  $(0, 1]$  in a natural way: represent a real number by its expansion in base 2. Rationals with denominator  $2^n$  in simplest form have two base 2 representations, one ending  $\dots 000\dots$  and the other ending  $\dots 1111\dots$ . Choose the one with infinitely many 1's: each real number in the interval  $(0, 1]$  has a unique binary representation with infinitely many 1's (including  $1 = 0.1111\dots$ ). Now associate this infinite binary representation with a set, containing  $n$  just in case  $\frac{1}{2^n}$  appears in the binary expansion. That is, we put 1 in the set if there is a 1 in the  $\frac{1}{2}$  place, 2 in the set if there is a 1 in the  $\frac{1}{2^2} = \frac{1}{4}$  place, and so on. In this way each real in  $(0, 1]$  is coded by a unique infinite set of natural numbers.]

Position 2 and position 3 are not mutually exclusive because one may have different attitudes about different systems. The more complex a construction, the more purely conventional decisions go into making it. It is easy to think of Frege natural numbers as *really* being the natural numbers; it is harder to convince oneself that one of the standard constructions of the real numbers (which differ from one another in purely technical ways) *really* gives the real numbers, because it is unclear which one it would be!

## 4 Propositional Functions

We now proceed to describe Russell's version of the construction of the number systems. Before we can do this, we have to look at his raw material. Russell didn't have ZFC (our modern set theory) to work with: he implements abstract objects as "propositional functions".

Consider for example the sentence " $x$  is an even prime". This is (for us as for Russell) an incomplete sentence, because  $x$  has not been assigned a definite reference. The usual term is "open sentence". Russell introduces a new notation " $\hat{x}$  is an even prime": this is no longer a sentence but notation for a *function* which takes objects of some implicitly understood type (it looks like natural numbers here) as input, and returns truth values (True or False) as output.

We give examples of "evaluation" of propositional functions.  $[\hat{x}$  is an even prime](2) evaluates to True and  $[\hat{x}$  is an even prime](17) evaluates to False, while  $[\hat{x}$  is an even prime]( $\frac{1}{2}$ ) is probably meaningless ( $\frac{1}{2}$  is not an object of the right type to ask this question about).

A propositional function with one variable can be thought of as a set: “ $\hat{x}$  is an even prime” would be the set  $\{2\}$  on this understanding. What it definitely is is a property, and we think of it as a set if we decide that the identity of a property depends only on which objects it is true of.

We can also have multi-place propositional functions, such as  $\hat{x} < \hat{y}$ , the “less-than” relation. This is distinguished from  $\hat{y} < \hat{x}$  by Russell’s convention that the letters which are first in the alphabet are supplied with values from the argument list first:

$$[\hat{x} < \hat{y}](4, 7) = 4 < 7 = \text{True}$$

while

$$[\hat{y} < \hat{x}](4, 7) = 7 < 4 = \text{False}$$

Notice that for Russell a relation is not as it is for us a set of ordered pairs: in fact, he defines ordered pairs as special relations rather than defining relations in terms of ordered pairs. One reason for this is that Russell was not aware of an implementation of the ordered pair as a set. Another reason might be his advocacy of Position 2: all of the definitions of the ordered pair are more or less artificial; there is no knockdown reason to choose one of them rather than another as *the* definition. Russell was not especially impressed when von Neumann showed him the first definition of the ordered pair as a set, which showed that all relation types could be implemented as types of sets.

Distinctions between kinds of object are important here. Every variable ranges over some particular kind of object, not over all objects whatsoever, and each new abstraction is of some particular kind determined by the kinds of its arguments.

3 is a natural number.

$\hat{x} < 3$  is a property or set of numbers (probably natural numbers).

$\hat{x} < \hat{y}$  is a two place relation on natural numbers.

$(\exists y. y \in \hat{x} \wedge (\forall z. z \in \hat{x} \rightarrow z = y))$  is a property of sets belonging to exactly those sets with one element. The type of the elements of these sets is not specified at this point. This is Russell’s number 1 (relative to the unspecified type of objects being counted).

Note that if sets are interpreted (as we suggest and as Russell actually does) as propositional functions, then  $x \in y$  abbreviates  $y(x)$ , the application of the propositional function  $y$  to  $x$ .

Attention to type is vitally important because we can otherwise produce Russell’s paradox with the greatest of ease in our new notation:

Suppose we can let  $x$  vary over *all* objects. Then we can form the sentence

$x(x)$  (apply  $x$  to itself), and from this the sentence  $\neg x(x)$ . From this we can form the propositional function  $\neg \hat{x}(\hat{x})$ , which we will call  $R$ . Note that  $R(x)$  means  $\neg x(x)$  for any  $x$ , including  $R$  itself, so  $R(R)$  is precisely equivalent to  $\neg R(R)$ , which is absurd! Notice that the propositional function  $R$  is precisely the set of all sets which are not elements of themselves if we interpret propositional functions of one argument as sets.

## 5 The construction of the positive rationals

We start with the natural numbers (which we may suppose already constructed as the Frege naturals:  $n$  is the set of all sets of objects of some unspecified basic type which have  $n$  elements in the usual sense).

The positive rational number usually written  $\frac{p}{q}$  is for us the relation  $\hat{x} : \hat{y} :: p : q$ , or  $\hat{x}q = \hat{y}p$ .

This is first of all quite natural: it is even intimately related to our Position 1 approach which doesn't talk about rationals at all!

If we regard relations as the same if they hold between the same natural numbers, then this gives the right simplifications of natural numbers.

For example,  $\frac{6}{4} = \frac{3}{2}$ , because the relation  $\hat{x} : \hat{y} :: 6 : 4$  is exactly equivalent to the relation  $\hat{x} : \hat{y} :: 3 : 2$ :

$$x : y :: 6 : 4 \leftrightarrow 4x = 6y \leftrightarrow 2x = 3y$$

(by a cancellation property for multiplication which holds for natural numbers)

$$\leftrightarrow x : y :: 3 : 2$$

The definitions of addition, multiplication, and order can then be the standard ones. There is a slight technical problem with these definitions: for example before one can use the definition  $(\frac{p}{q})(\frac{r}{s}) = \frac{pr}{qs}$  one needs to show that writing  $\frac{p}{q}$  or  $\frac{r}{s}$  as ratios of different natural numbers will give an equivalent product.

## 6 The construction of the positive reals

Now we have the "line" of positive rationals, which we know has "holes" in it, such as the "hole" at  $\sqrt{2}$ . Russell does not allow us to simply postulate something to put in the "hole": the whole point is that there is nothing at

all in the hole! The actual objects we can construct which are related to the “hole” in the positive rationals are the pair of sets on either side of it:  $\hat{p}^2 < 2$  and  $\hat{p}^2 > 2$  are the propositional functions which implement these sets (where  $p$  is a rational variable). These two sets make up what Dedekind called a “cut” (Dedekind constructed real numbers as *pairs* of sets of general rational numbers (not just positive rational numbers)). Russell notes that he only needs the segment below the “cut”, so he identifies the square root of 2 with  $\hat{p}^2 < 2$ , or  $\{r \in \mathcal{Q} \mid r^2 < 2\}$ , in more familiar set notation. For Russell, a positive real number is an initial segment in the positive rationals.

A segment in the positive rationals cannot be defined in terms of its upper endpoint (which may not exist yet), but can be defined using the following criteria:

A set  $r$  of positive rationals is a positive real number (for Russell) just in case it satisfies three conditions:

**nontriviality:** It is not either the empty set or the set of all positive rationals.

**downward closure:** For any rationals  $p$  and  $q$ , if  $p \in r$  and  $q < p$ , then  $q \in r$ .

**no maximum:** For any element  $p$  of  $r$ , there is an element  $q$  of  $r$  such that  $q \geq p$ .

The last condition is a technicality: it ensures that a positive rational such as 1 is represented uniquely: on the basis of the other two conditions alone, both  $\hat{p} < 1$  and  $\hat{p} \leq 1$  would be positive reals, and they would both appear to represent 1: we exclude the latter set with the third condition, since it has a largest element.

Addition, multiplication and order have simple and natural definitions.

$r < s$  means  $r \subset s$

$r + s = \{r' + s' \mid r' \in r \wedge s' \in s\}$

$rs = \{r's' \mid r' \in r \wedge s' \in s\}$

The definition of multiplication is more complicated if cuts of the general rationals are used to define the reals.

## 7 The construction of the general reals

We can define these in the same way that Russell defines the integers.

Define  $+r$  as the function on general reals which takes a real  $s$  to the real  $r + s$ : this can be written as the relation  $\hat{y} = \hat{x} + r$ . Similarly, define  $-r$  as the function  $\hat{y} = \hat{x} - r$  (or equivalently  $\hat{y} + r = \hat{x}$ , and  $0$  as the function  $\hat{y} = \hat{x}$ .

[It is an amusing pun that equality on natural numbers is the same object for Russell as the positive rational 1, while equality on positive reals is the same object for Russell as the real number 0. Unexpected identifications like these have been used in arguments for the conventional nature of these constructions.]

This construction can be made to look more like the construction of the positive rationals from the natural numbers, as follows: for any positive reals  $r$  and  $s$ , define  $r - s$  as the relation  $\hat{x} + s = \hat{y} + r$ . A little thought shows that this definition is actually the same as the definition above (the same propositional functions are identified with the same general reals). Expected equations (such as  $2 - 1 = 3 - 2$ ) are verified by showing that the relations they represent hold between the same pairs of objects.

## 8 A common feature

All of these constructions share the characteristic that the new number system at each stage of the construction does not contain the old number system. The natural number 1 is not the same object as the rational  $\frac{1}{1}$ , nor is it the same object as the positive real 1 (verify that the positive rational  $\frac{1}{1}$  is the equality relation on the natural numbers, while the positive real 1 is the set of all proper fractions, both rather unexpected identifications).

Russell's type system actually prevents the various systems from being nested in the way we usually expect them to be. The fact that the systems have different functions might reconcile us to this phenomenon: if the counting number 1 is not the same object as the positive rational 1 or the integer 1, this might help to discourage us from trying to use inappropriate positive rationals (such as  $\frac{1}{2}$ ) or integers (such as  $-1$ ) to count... But even in standard set theory, where we *could* have them nested, this does not happen – except in Conway's system of *surreal numbers*, in which all the real numbers (and quite a few more numbers!) are constructed *all at once!* We may look at this implementation later.

## 9 The complex numbers

We will present three different definitions of the complex numbers (as we did in class) and compare them with respect to some philosophical criteria.