

# Desirable features of axioms and undefined notions

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In this handout I list and discuss some desirable features of systems of axioms (some of which also apply to undefined notions). If anyone thinks of any points from the class discussion which I left out, feel free to bring them to my attention.

**Consistency:** This was brought up by a student – it is of course interesting and follows from “soundness” below.

**Completeness:** We would like to be able to prove all true theorems, or at least we would like to be able to prove all the theorems we know to be true (all the theorems in our math textbook for the particular subject, perhaps). [We will see later on that the standard axioms for arithmetic are incomplete in the sense that they cannot prove all true statements about arithmetic, but nonetheless they do allow us to prove all “natural” theorems of arithmetic.]

**Soundness:** We would like the axioms to be true in the intended interpretation. Notice that if the axioms of a theory are true in its intended interpretation, they will certainly be consistent. If a theory has a well-understood intended interpretation, the previous point of *completeness* of the axioms might be less vital: one can appeal to “intuition” about the intended (and presumably familiar) subject matter to supplement gaps in the axioms. This is why Euclid’s geometry works, even though many needed axioms are not explicitly given.

Hilbert has to be much more careful about completeness of his axioms: he is less comfortable with the idea that geometric truth can be backed

up with our spatial intuition, in the aftermath of non-Euclidean geometry and general relativity. He is also working in the aftermath of the crisis in foundations caused by the discovery of paradoxes in set theory (we'll look at these later) which led mathematicians working in foundations to be extremely careful about logic. As one of you pointed out, Hilbert wanted his axioms to be sufficiently complete that one could change the names of all the undefined concepts in a way which would break the connection with intuition, and yet still be able to follow the proofs of the theorems.

Hilbert's geometry does have an intended interpretation, in spite of the effort to make the proofs rely as little as possible (ideally not at all) on intuition. Many modern axiomatic theories do not have an intended interpretation: such theories characterize classes of structures (the axioms of a group were given in class as an example of this). In such a theory one needs to be very careful about any appeal to intuition, as intuition about such a theory may derive from your knowledge of particular examples which have special properties which do not follow from the general axioms (for example, most groups familiar to you satisfy commutativity of the operation, which does not hold in general).

**Independence (economy):** A nice feature of a set of axioms (or set of undefined notions) is "independence": one would like it to be the case that no axiom can be proved from the other axioms, and that no undefined notion can in fact be defined using the other undefined notions.

This is a criterion of economy: we would like to have as few axioms and undefined notions as possible. This criterion can conflict with other desirable features of axiom sets.

I gave the extended example of undefined notions of propositional logic. "and", "or", and "not" make up a natural set of primitives: it seems reasonable to use these to define implication and equivalence.

It is actually the case that "and" and "not" are sufficient, and that "or" and "not" are sufficient. The resulting definition of the third connective is rather cumbersome, but this is not the most interesting objection (once a definition is made, one can use the new notation it introduces and not worry about the way it is defined except perhaps in the proofs of a few basic theorems about the defined notion). A

much more interesting objection is that the full set of three connectives is more natural because it is symmetric: there is no reason to prefer defining “and” in terms of “or” and “not” as opposed to defining “or” in terms of “and” and “not”: the two definitions have basically the same structure, and the two concepts have the same level of complexity.

A further example: the connective “neither  $P$  nor  $Q$ ” can be used to define all the other connectives. But it doesn’t seem better to work with just this connective: it may seem less basic or natural than the connectives in larger sets of primitive notions.

Similar considerations apply to axiom sets. When we make a set of axioms as small as possible, it may turn out that we start our mathematical development by giving very hard proofs of many obvious results where it might seem more natural to introduce the results as axioms.

Another possible disadvantage of a strong focus on economy is that it may make errors more likely. Axioms and undefined notions formulated in such a way as to be as economical as possible may be unnatural assertions or unnatural as primitives (in the way that “neither” seems less basic than “and”): the formulation of such axioms, which are farther from being “self-evident” may be more likely to be incorrect. A very economical system of axioms necessitates hard proofs of easy results at the beginning of the subject: in a hard proof we may make mistakes, and if the theorem to be proved is “obviously” true, we may be more likely to overlook our errors (since the notions we are using to construct the proof are farther from “intuition” than the result we are proving!)

### **Technical convenience; naturalness :**

These are two different (but related) criteria which can conflict with the criteria of independence and economy.

Technical convenience: we would like it to be easy to state and prove the basic theorems of the subject. Both the form of definitions and the choice of axioms are often motivated (especially in small technical details) by the desire to make it easier to prove important theorems.

Naturalness: we would like the axioms and definitions to follow the natural conceptual structure of the subject. (this is related to the classical criterion of “self-evidence”: among other things, we would

like proofs to proceed from obviously true statements to less obviously true statements).

An example of failure of naturalness, to my mind, is the need to use Pasch's axiom (II, 4) to prove that there is a point between any two points in Hilbert's axiom scheme. On esthetic grounds and even certain practical grounds, it would be nice to have a self-contained theory of the geometrical line (this can be done, and I may demonstrate how to do this in class later on). If one were to develop a Hilbert geometry of the line (which I might do), one would need to introduce more axioms of betweenness, because one then could not use Pasch's axiom to fill in the gaps...

I'll look for examples of the technical convenience criterion as we go along.

**Conflicts between the criteria:** It seems to me that the criterion of economy is likely to conflict with the connected criteria of naturalness and independence, as I have discussed. Any other suggestions?