Rules Handout for Test I

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Contents

1 Natural deduction

1.1 Rules for conjunction

Conjunction introduction:

$$\frac{\begin{array}{r} P \\ Q \end{array}}{P \wedge Q}$$

Conjunction elimination:

$$\frac{P \wedge Q}{P}$$

or

$$\frac{P \wedge Q}{Q}$$

In each rule, the items above the line are premises (which can be taken from earlier lines of proofs). The item below the line is the conclusion (a new line for the proof).

The rules can be abbreviated c.i. and c.e.; the two different forms of c.e. could be called c.e. 1 and c.e. 2, but this seems too pedantic.

1.2 Rules for implication

Implication elimination (modus ponens):

$$\frac{\begin{array}{r} P \\ P \to Q \end{array}}{Q}$$

This rule may be abbreviated either i.e. or m.p. according to taste.

Implication introduction: This rule has a rather different structure.

1. P hypothesis for implication introduction Goal: Q ... (representing an unknown number of proof lines) n. Q ??? (we don't know what rule we use to get the conclusion)

n+1. $P \to Q$ implication introduction, lines 1-n.

The box indicates a "subproof" with the additional hypothesis P. No line of the subproof can be used outside the subproof. Lines outside the subproof may be used in the subproof unless they belong to some other subproof which does not contain the given subproof.

This rule embodies the natural strategy for proving an implication $P \to Q$: assume P and prove Q; this proof allows you to conclude that $P \to Q$ is true whether P is true or not (if P is true, we use the auxiliary proof to show that Q is also true, so $P \to Q$ is true; if P is false, $P \to Q$ is vacuously true.)

The line labelled “Goal:” is a comment: it tells us what conclusion we need to close the subproof.

1.3 Rules for negation

We resume the development of rules for the other logical operations.

negation elimination:

$$\frac{\begin{array}{r} P \\ \neg P \end{array}}{Q}$$

This expresses the idea that anything can be deduced from a contradiction (any proposition is implied by a false proposition, as is clear from the truth table for implication).

negation introduction: This is another rule of the same form as implication introduction.

1. P hypothesis for negation introduction Goal: contradiction ... (representing an unknown number of proof lines) n. $Q \wedge \neg Q$??? (we don’t know what rule we use to get the conclusion)

n+1. $\neg P$ negation introduction, lines 1-n.

We prove that P is false by assuming P and deducing a contradiction.

1.4 Rules for disjunction

Each of the rules for disjunction is like the rule of conjunction elimination in having two symmetric forms. We will not require use of separate names for the two forms of each rule.

Disjunction elimination: Either

$$\frac{\begin{array}{r} P \vee Q \\ \neg P \end{array}}{Q}$$

or

$$\begin{array}{r} P \lor Q \\ \neg Q \\ \hline P \end{array}$$

Disjunction introduction: Either

> 1. $\neg P$ hypothesis for disjunction introduction
> Goal: Q
> … (representing an unknown number of proof lines)
> n. Q ??? (we don't know what rule we use to get the conclusion)

n+1. $P \lor Q$ disjunction introduction, lines 1-n.

or

> 1. $\neg Q$ hypothesis for disjunction introduction
> Goal: P
> … (representing an unknown number of proof lines)
> n. P ??? (we don't know what rule we use to get the conclusion)

n+1. $P \lor Q$ disjunction introduction, lines 1-n.

1.5 Rules for the biconditional

For now, our official rules for the biconditional amount to a definition of $P \leftrightarrow Q$ as $(P \to Q) \land (Q \to P)$:

Biconditional introduction:

$$\begin{array}{r} (P \to Q) \land (Q \to P) \\ \hline P \leftrightarrow Q \end{array}$$

Biconditional elimination:

$$\begin{array}{r} P \leftrightarrow Q \\ \hline (P \to Q) \land (Q \to P) \end{array}$$

After the due date of the first homework set, I will post another set of rules for the biconditional.

1.6 Derived rules

The rules already given are the basic rules of our system, but of course there are many simple rules of inference that are equally basic to reasoning in practice. In this subsection, we derive some of them.

Excluded middle :

Goal: $P \vee \neg P$

> 1. $\neg P$ hypothesis for disjunction introduction.
> 2. $\neg P$ copy line 2.

3. $P \vee \neg P$ disjunction introduction, lines 1-2.

We can use this to prove any statement of the form $P \wedge \neg P$, and rather than reproduce this proof whenever we do this, we use the new rule

$$\overline{P \vee \neg P}$$

which we call "excluded middle" (or exc. mid.). Note that this rule does not require any premises.

Double negation: Under this heading, we prove the rules for introduction and elimination of double negations, and introduce the method of proof by contradiction.

Double negation introduction :

1. P

Goal: $\neg\neg P$

> 2. $\neg P$ hypothesis for negation intro
> 3. $P \wedge \neg P$ c.i., lines 1,2.

4. $\neg\neg P$ negation intro, lines 2-3.

This justifies the rule of "double negation introduction (d.n.i.)":

$$\frac{P}{\neg\neg P}$$

Double negation elimination :

1. $\neg\neg P$

Goal: P

2. $P \vee \neg P$ exc. mid.

3. P d.i., lines 1,2.

This justifies the rule of "double negation elimination (d.n.e.)":

$$\frac{\neg\neg P}{P}$$

Proof by contradiction :

Goal: P

> 1. $\neg P$ hypothesis for n.i.
>
> . . .
>
> n. $Q \wedge \neg Q$

n+1. $\neg\neg P$ negation intro, lines 1-n

n+2. P, d.n.e., line n+1

justifies the rule of "proof by contradiction" (which is just one line shorter):

Goal: P

> 1. $\neg P$ hypothesis for proof by contradiction
>
> . . .
>
> n. $Q \wedge \neg Q$

n+1. P, proof by contradiction, lines 1-n

Modus tollens: The rule of *modus tollens* (m.t.) is the additional implication elimination rule

$$\frac{\begin{array}{r} P \to Q \\ \neg Q \end{array}}{\neg P}$$

It is justified by the following proof:

1. $P \to Q$

2. $\neg Q$

Goal: $\neg P$

> 3. P hypothesis for n.i.
> 4. Q m.p., lines 1,3
> 5. $Q \wedge \neg Q$ c.i., lines 2,4

6. $\neg P$ n.i., lines 3-5.

Simple disjunction introduction: The rules of *simple disjunction introduction* are

$$\frac{P}{P \wedge Q}$$

and

$$\frac{Q}{P \wedge Q}$$

The formal proofs follow:

1. P

Goal: $P \vee Q$

> 2. $\neg Q$ hypothesis for disjunction intro
> Goal: P
> 3. P copy line 1.

4. $P \vee Q$ d.i. lines 2-3.

The other proof is very similar:

1. Q

Goal: $P \vee Q$

> 2. $\neg P$ hypothesis for disjunction intro
> Goal: Q
> 3. Q copy line 1.

4. $P \vee Q$ d.i. lines 2-3.

The simple disjunction rules (s.d.i.) are often presented as the basic disjunction introduction rules. In this case, one will also need reasoning by cases (the next derived rule) as an elimination rule, and excluded middle. Our choice of basic disjunction rules is more economical.

Reasoning by cases: The rule of "reasoning by cases" ("cases" for short) is

$$\begin{array}{r} P \vee Q \\ P \rightarrow R \\ Q \rightarrow R \\ \hline R \end{array}$$

The formal proof follows:

1. $P \vee Q$

2. $P \rightarrow R$

3. $Q \rightarrow R$

Goal: R

> 4. $\neg R$ hypothesis for proof by contradiction
> 5. $\neg P$ m.t., lines 2,4
> 6. Q d.e., lines 1,5
> 7. R m.p., lines 3,6
> 8. $R \wedge \neg R$, c.i. lines 4,7

9. R proof by contradiction, lines 4-8.

de Morgan rules

The de Morgan rules are as follows:

$$\frac{\neg(P \vee Q)}{\neg P \wedge \neg Q}$$

1. $\neg(P \vee Q)$ premise

Goal: $\neg P \wedge \neg Q$

Goal: $\neg P$

> 2. P hyp for n.i.
>
> 3. $P \vee Q$ s.d.i, line 2.
>
> 4. $(P \vee Q) \wedge \neg(P \vee Q)$ c.i. 1,4

5. $\neg P$ n.i., lines 2-4.

Goal: $\neg Q$

> 6. Q hyp for n.i.
>
> 7. $P \vee Q$ s.d.i, line 6.
>
> 8. $(P \vee Q) \wedge \neg(P \vee Q)$ c.i. 1,7

9. $\neg Q$ n.i., lines 6-8.

10. $\neg P \wedge \neg Q$ c.i. 5,9

$$\frac{\neg P \wedge \neg Q}{\neg(P \vee Q)}$$

1. $\neg P \wedge \neg Q$ premise

Goal: $\neg(P \vee Q)$

> 2. $P \vee Q$ hyp. for n.i.
>
> 3. $\neg P$ c.e., line 1
>
> 4. $\neg Q$ c.e., line 1
>
> 5. P d.e., lines 2,4
>
> 6. $P \wedge \neg P$ c.i., lines 5,3.

7. $\neg(P \vee Q)$ n.i., lines 2-6.

$$\frac{\neg(P \wedge Q)}{\neg P \vee \neg Q}$$

1. $\neg(P \wedge Q)$ premise

2. Goal: $\neg P \vee \neg Q$

> 3. $\neg\neg P$ hyp for d.i.
>
> Goal: $\neg Q$
>
> > 4. Q hyp for n.i.
> >
> > 5. P d.n.e. line 3.
> >
> > 6. $P \land Q$ c.i. lines 5,4.
> >
> > 7. $(P \land Q) \land \neg(P \land Q)$ c.i., lines 1,6.
>
> 8. $\neg Q$ n.i., lines 4-7.

9. $\neg P \lor \neg Q$ d.i. lines 3-8.

$$\frac{\neg P \lor \neg Q}{\neg(P \land Q)}$$

1. $\neg P \lor \neg Q$ premise

Goal: $\neg(P \land Q)$

> 2. $P \land Q$ hyp for n.i.
>
> 3. P c.e., line 2.
>
> 4. $\neg\neg P$ d.n.i, line 3.
>
> 5. $\neg Q$ d.e., lines 4,1.
>
> 6. Q c.e., line 2.
>
> 7. $Q \land \neg Q$ c.i., lines 5,6.

8. $\neg(P \land Q)$ n.i., lines 2-7.

2 Boolean algebra

I list the axioms of "boolean algebra".

$$\begin{array}{cc}
p+q=q+p & pq=qp \\
(p+q)+r=p+(q+r) & (pq)r=p(qr) \\
p+p=p & pp=p \\
p+(-p)=1 & p(-p)=0 \\
p+0=p & p1=p \\
p+1=1 & p0=0 \\
p(q+r)=pq+pr & p+qr=(p+q)(p+r) \\
-(p+q)=(-p)(-q) & -(pq)=(-p)+(-q)
\end{array}$$

$$--p=p$$

If we interpret 0 as F, 1 as T, $-p$ as $\neg p$, pq as $p \wedge q$, and $p+q$ as $p \vee q$, all the axioms are interpreted as logical equivalences. All of the equivalences used in the previous section are found here, plus some extras.

$p \rightarrow q \equiv \neg p \vee q$, so we translate it to $-p+q$.

3 Mod 2 arithmetic

$$\begin{array}{ccc} x+y=y+x & & xy=yx \\ (x+y)+z=x+(y+z) & & (xy)z=x(yz) \\ x+0=x & & x1=x \\ & x(y+z)=xy+xz & \end{array}$$

There are two additional special rules which clearly hold:

$$x+x=0 \qquad xx=x$$

In the lecture in class I included $x0=x$ as an axiom, but in fact $x0=x(y+y)=xy+xy=0$ proves this from the axioms given.

If we interpret 1 as true and 0 as false, then multiplication will be interpreted as $\wedge$ and addition will be interpreted as $\oplus$, the "exclusive or".

We now show how to define other logical connectives in terms of this algebraic interpretation. $\neg x = 1+x$ is easily checked.

$$x \vee y = \neg(\neg x \wedge \neg y) = 1+(1+x)(1+y) = 1+1+x+y+xy = x+y+xy$$

establishes the definition of $x \vee y = x+y+xy$.

$$x \rightarrow y = \neg x \vee y = \neg x + y + (\neg x)y = (1+x)+y+(1+x)y$$

$$= 1+x+y+y+yx = 1+x+xy$$

establishes the definition $x \rightarrow y = 1+x+xy$.

$$x \leftrightarrow y = (x \rightarrow y)(y \rightarrow x) = (1+x+xy)(1+y+yx)$$

$$= 1+y+yx+x+xy+xyx+xy+xyy+xyyx = 1+x+y$$

(in the last step, the xy terms cancel in pairs).

4 Sequent calculus

We say that a sequent $\Gamma \vdash \Delta$ is *valid* if any assignment of truth values to propositional letters in the sequent which makes *all* propositions in the set Γ true makes *some* proposition in the set Δ true.

A simple example of a valid sequent is $P \vdash P$. More generally, any sequent $\Gamma, P \vdash P, \Delta$ in which there is a proposition which appears in both sets is valid: if all propositions on the left are true, then P is true, and P is one of the propositions on the right as well.

We introduce basic rules for negation and conjunction, and use them to derive rules for disjunction and implication.

$$\frac{\Gamma \vdash A, \Delta}{\Gamma, \neg A \vdash \Delta}$$

$$\frac{\Gamma, A \vdash \Delta}{\Gamma \vdash \neg A, \Delta}$$

The negation rules simply cause the negated expression to be moved across the "turnstile".

To see (for example) that the first rule is valid, suppose that the sequent above the bar is valid, so that any assignment of truth values to propositional letters which makes all sentences in Γ true either makes A true or makes some sentence in Δ true. This implies that any assignment of truth values to propositional letters which makes all sentences in Γ true and also makes $\neg A$ true must make some sentence in Δ true, which is the condition of validity of the sequent below the bar.

$$\frac{\Gamma, A \wedge B \vdash \Delta}{\Gamma, A, B \vdash \Delta}$$

$$\frac{\Gamma \vdash A, \Delta \qquad\qquad \Gamma \vdash B, \Delta}{\Gamma \vdash A \wedge B, \Delta}$$

We present the derivation of an obviously valid sequent.

$$\begin{array}{c} P \vdash P \\ \hline P, \neg P \vdash \\ \hline P \wedge \neg P \vdash \\ \hline \vdash \neg(P \wedge \neg P) \end{array}$$

A sequent $\vdash P$ asserts that P is a tautology (if every proposition in the empty set is true (vacuously true) then P is true). Similarly, a sequent $P \vdash$ asserts that P is contradictory. The sequent with which we start is valid because it has P on both sides. The sequent is built from the bottom: an advantage of the sequent calculus is that the construction of proofs is driven by the form of the statement to be proved (this is partially true in natural deduction proofs).

We now present the derivation of rules for disjunction and implication.

The derivation

$$\dfrac{\dfrac{\dfrac{\dfrac{\Gamma, P \vdash \Delta}{\Gamma \vdash \neg P, \Delta} \qquad\qquad \dfrac{\Gamma, Q \vdash \Delta}{\Gamma \vdash \neg Q, \Delta}}{\Gamma \vdash \neg P \wedge \neg Q, \Delta}}{\Gamma, \neg(\neg P \wedge \neg Q) \vdash \Delta}}{\Gamma, P \vee Q \vdash \Delta}$$

justifies the rule

$$\dfrac{\Gamma, P \vdash \Delta \qquad\qquad \Gamma, Q \vdash \Delta}{\Gamma, P \vee Q \vdash \Delta}$$

The derivation

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\Gamma \vdash P, Q, \Delta}{\Gamma, \neg P \vdash Q, \Delta}}{\Gamma, \neg P, \neg Q \vdash \Delta}}{\Gamma, \neg P \wedge \neg Q \vdash \Delta}}{\Gamma \vdash \neg(\neg P \wedge \neg Q)}}{\Gamma \vdash P \vee Q, \Delta}$$

justifies the rule

$$\dfrac{\Gamma \vdash P, Q, \Delta}{\Gamma \vdash P \vee Q, \Delta}$$

Notice the symmetry between the rules for the dual operations $\wedge$ and $\vee$; it is not accidental.

The derivation

$$\begin{array}{c} \Gamma, P \vdash Q, \Delta \\ \hline \Gamma \vdash \neg P, Q, \Delta \\ \hline \Gamma \vdash \neg P \vee Q, \Delta \\ \hline \Gamma \vdash P \rightarrow Q, \Delta \end{array}$$

justifies the rule

$$\frac{\Gamma, P \vdash Q, \Delta}{\Gamma \vdash P \rightarrow Q, \Delta}$$

Set $\Delta = \emptyset$ in this rule and get

$$\frac{\Gamma, P \vdash Q}{\Gamma \vdash P \rightarrow Q}$$

If you read this carefully, you will see that this is basically the implication introduction rule of our natural deduction system.

The derivation

$$\dfrac{\dfrac{\Gamma \vdash P, \Delta}{\Gamma, \neg P \vdash \Delta} \qquad\qquad \Gamma, Q \vdash \Delta}{\begin{array}{c} \Gamma, \neg P \vee Q \vdash \Delta \\ \hline \Gamma, P \rightarrow Q \vdash \Delta \end{array}}$$

justifies the rule

$$\frac{\Gamma \vdash P, \Delta \qquad\qquad \Gamma, Q \vdash \Delta}{\Gamma, P \rightarrow Q \vdash \Delta}$$

This rule will look quite peculiar, understandably. We do not usually use $P \rightarrow Q$ as a premise all by itself; we use it in the context of a rule like *modus ponens* with an additional hypothesis.