I decided against doing rewrites of Homework 4, but I do think you need some further work. I am giving you this practice test and some additional exercises (the practice test part will look roughly the way I imagine your test will; the additional exercises are things it is good for you to do, and individual things of this kind might appear on the test).

This worksheet is due on Friday. I will post solutions to it on Thursday so you can self-mark it. It will be marked simply on the basis that it is turned in. I may look at papers and comment, and since you will have seen the solutions you will be able to ask questions intelligently on Friday.

We will be covering new material in lecture this week, and new homework will be posted early this week, due after the test. I will also be spending some time on review, notably on doing problems that I marked on Homework 4.

Test II will be given on Monday, April 11.

It is possible that I will add more problems to this document.
1 Practice Test II

The idea of this section is that it might look like your actual test in length and difficulty.

1. Prove one of these using the axioms from our book (and additional statements as the instructions permit). If you prove more than one, your best work will count and you may receive additional credit.

(a) For any \( x \), \( x0 = 0 \) (strictly from the axioms, with no “algebra” justifications of steps at all).

**Solution:**

i. \( x0 = x \cdot (0 + 0) \) identity of addition
ii. \( x \cdot 0 = x \cdot 0 + x \cdot 0 \) distributivity
iii. \( x \cdot 0 + -(x \cdot 0) = (x \cdot 0 + x \cdot 0) + -x \cdot 0 \) add same thing to both sides
iv. \( 0 = x \cdot 0 + (x \cdot 0 + -(x \cdot 0)) \) inverse property of addition on left, associativity of addition on right
v. \( 0 = x \cdot 0 + 0 \) inverse property of addition
vi. \( 0 = x \cdot 0 \) identity of addition: this is what we wanted to prove.

(b) If \( xy = 0 \) then \( x = 0 \) or \( y = 0 \). (you may use the theorem that for all \( x \), \( x0 = 0 \); otherwise just the axioms).

**Assume:** \( xy = 0 \)

**Goal:** \( x = 0 \) or \( y = 0 \)

**Assume:** \( \neg x = 0 \)

**Goal:** \( y = 0 \)

**Proof:**

i. \( xy = 0 \) by assumption above
ii. \( x^{-1}(xy) = x^{-1}0 \) multiply left side of both by same thing; \( x^{-1} \) is defined because we have assumed \( x \neq 0 \)
iii. \( (x^{-1}x)y = 0 \) associativity of mult on left; zero property on right
iv. \( 1y = 0 \) inverse property of mult and assumption \( x \neq 0 \)
v. \( y = 0 \) identity of mult; this is our goal.
(c) For any \( x, x^2 \geq 0 \). You may use the theorems \( x0 = x, (-x)(-y) = xy \) and \((-x)y = -xy\); otherwise you may use only the axioms (you may use the alternative axioms for order if you prefer). If you use the alternative axioms for order, then \( x \in P \) is defined as \( x > 0 \), not the other way around.

Solution:

**Version using (P10-P12)** By P10 one of three things is true, \( x \in P, x = 0, \) or \(-x \in P\). We prove by cases.

- **Case 1** assume \( x \in P \). Then by P13 \( x^2 = x \cdot x \in P \) so \( x^2 > 0 \), so \( x^2 \geq 0 \).
- **Case 2** assume \( x = 0 \). Then \( x^2 = 0 \cdot 0 = 0 \) so \( x^2 = 0 \) so \( x^2 \geq 0 \).
- **Case 3** assume \(-x \in P\). Then \((-x)(-x) \in P \) by P13. But \((-x)(-x) = xx = x^2 \), so \( x^2 \in P \), so \( x^2 > 0 \).

The goal \( x^2 \geq 0 \) has been proved in each case.

**Version using (P10’)-(P13’)** By P10’ we have that one of \( x > 0, x = 0, \) or \( 0 > x \) is true. We prove by cases.

- **Case 1**: Assume \( x > 0 \). Because \( x > 0 \) we can by (P13’) multiply both sides of this by \( x \), getting \( x \cdot x > 0 \cdot x = 0 \), so \( x^2 > 0 \), so \( x^2 \geq 0 \).
- **Case 2**: Assume \( x = 0 \). We then have \( x^2 = 0 \cdot 0 = 0 \), so \( x^2 = 0 \) so \( x^2 \geq 0 \).
- **Case 3** (pay attention to this one); Assume \( 0 > x \). Use (P12’) to add \(-x\) to both sides, so \( 0 + -x > x + -x \), so \(-x > 0 \) (inverse and identity properties of addition). Since \(-x > 0 \) we can multiply both sides of this inequality by \(-x\): \((-x)(-x) > (-x)0\), so \( x^2 > 0 \), so \( x^2 \geq 0 \).

The goal \( x^2 \geq 0 \) has been proved in each case.
2. Prove that \( \lim_{x \to 2} x^3 = 8 \), by showing how to find for each \( \epsilon > 0 \) a \( \delta > 0 \) such that if \( 0 < |x - 2| < \delta \) then \( |x^3 - 8| < \epsilon \). Your work should include a proof that your \( \delta \) works, not just the scratch work to find it.

**scratch work:** We want to make \( |x^3 - 8| < \epsilon \) by making \( |x - 2| \) small.

\[
|x^3 - 8| = |x - 2||x^2 + 2x + 4| \leq |x - 2|(|x|^2 + 2|x| + 4)
\]

we need an upper bound on \( |x|^2 + 2|x| + 4 \). If we assume \( |x - 2| < 1 \), we get \( 1 < x < 3 \), so \( x^2 + 2x + 4 < 19 \).

\[
|x^3 - 8| = |x - 2||x^2 + 2x + 4| \leq |x - 2|(|x|^2 + 2|x| + 4) < |x - 2|(19),
\]

and this will be less than \( \epsilon \) if \( |x - 2| < \frac{\epsilon}{19} \).

**Proof:** Let \( \epsilon > 0 \) be arbitrary.

Let \( \delta = \min(1, \frac{\epsilon}{19}) \).

Let \( x \) be arbitrary. Assume \( 0 < |x - 2| < \delta = \min(1, \frac{\epsilon}{19}) \).

Notice that this means that \( |x - 2| < 1 \) so \( 1 < x < 3 \).

Now \( |x^3 - 8| = |x - 2||x^2 + 2x + 4| < |x - 2|(19) \) (because \( x < 3 \)) \( < \frac{\epsilon}{19}(19) = \epsilon \), which is what we needed to prove.
3. Prove that if \( \lim_{x \to a} f(x) = L \) and \( \lim_{x \to a} g(x) = M \) then

\[
\lim_{x \to a} (f(x) - g(x)) = L - M.
\]

**Scratch work:**

We want to make \(|(f(x) - g(x)) - (L - M)| < \epsilon\), by controlling the size of \(|f(x) - L|\) and \(|g(x) - M|\).

\[
|(f(x) - g(x)) - (L - M)| = |(f(x) - L) + (M - g(x))| \leq |f(x) - L| + |M - g(x)| = |f(x) - L| + |g(x) - M|.
\]

To make this less than \(\epsilon\), make each of \(|f(x) - L|\) and \(|g(x) - M|\) less than \(\frac{\epsilon}{2}\).

**Proof:**

Assume (1) \( \lim_{x \to a} f(x) = L \) and (2) \( \lim_{x \to a} g(x) = M \).

Let \( \epsilon > 0 \) be arbitrary.

Choose \( \delta_1 \) such that for any \( x \), if \( 0 < |x - a| < \delta_1 \) then \( |f(x) - L| < \frac{\epsilon}{2} \).

Choose \( \delta_2 \) such that for any \( x \), if \( 0 < |x - a| < \delta_2 \) then \( |g(x) - M| < \frac{\epsilon}{2} \).

Let \( \delta = \min(\delta_1, \delta_2) \).

Let \( x \) be arbitrary.

Assume \( 0 < |x - a| < \delta \).

Now \(|(f(x) - g(x)) - (L - M)| = |(f(x) - L) + (M - g(x))| \leq |f(x) - L| + |M - g(x)| = |f(x) - L| + |g(x) - M| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon\). This is what we needed to show.
4. Prove one of the following statements of your choice. If you prove both you will receive additional credit.

(a) Suppose that \( \lim_{x \to a} f(x) = \lim_{x \to a} h(x) = L \) and for all \( x \) we have \( f(x) \leq g(x) \leq h(x) \). Show that \( \lim_{x \to a} g(x) = L \).

**Solution:**

Let \( \epsilon > 0 \) be arbitrary.

Choose \( \delta_1 \) so that for any \( x \), if \( 0 < |x - a| < \delta_1 \), then \( |f(x) - L| < \epsilon \).

Choose \( \delta_2 \) so that for any \( x \), if \( 0 < |x - a| < \delta_2 \), then \( h(x) - L| < \epsilon \).

Let \( \delta = \min(\delta_1, \delta_2) \).

Let \( x \) be arbitrary.

Suppose \( 0 < |x - a| < \delta \).

Recall that \( |a - b| < c \) is equivalent to \( -c < a - b < c \) and so to \( b - c < a < b + c \): so for example \( |f(x) - L| < \epsilon \) is equivalent to \( L - \epsilon < f(x) < L + \epsilon \).

It follows that we have \( L - \epsilon < f(x) < L + \epsilon \) and \( L - \epsilon < h(x) < L + \epsilon \). Putting this together with what we know about \( g(x) \), we get \( L - \epsilon < f(x) < g(x) < h(x) < L + \epsilon \), so \( L - \epsilon < g(x) < L + \epsilon \), that is, \( |g(x) - L| < \epsilon \), which is what we needed to show.
(b) Suppose that \( f(x) \leq g(x) \) for all \( x \) and that both \( \lim_{x \to a} f(x) \) and \( \lim_{x \to a} g(x) \) exist. Show that \( \lim_{x \to a} f(x) \leq \lim_{x \to a} g(x) \).

**Solution:**

Suppose that \( f(x) \leq g(x) \) for all \( x \).

Suppose that \( \lim_{x \to a} f(x) = L \) and \( \lim_{x \to a} g(x) = M \).

Suppose for the sake of a contradiction that \( L > M \) (so \( L - M \) is positive).

Choose \( \delta_1 \) so that for any \( x \), if \( 0 < |x - a| < \delta_1 \) then \( |f(x) - L| < \frac{L-M}{2} \).

Choose \( \delta_2 \) so that for any \( x \), if \( 0 < |x - a| < \delta_2 \) then \( |g(x) - M| < \frac{L-M}{2} \).

so \( L - \frac{L-M}{2} < f(x) < L + \frac{L-M}{2} \)

and \( M - \frac{L-M}{2} < g(x) < M + \frac{L-M}{2} \).

But this means that \( g(x) < M + \frac{L-M}{2} = \frac{L+M}{2} = L - \frac{L-M}{2} < f(x) \),

so \( g(x) < f(x) \), which is a contradiction.

Give an example of functions \( f \) and \( g \) such that \( f(x) < g(x) \) for all \( x \) and \( \lim_{x \to 0} f(x) = \lim_{x \to 0} g(x) \).

**Solution:**

Let \( f(x) = -x^2 \) for \( x \neq 0 \) and \( f(0) = -1 \), and \( g(x) = x^2 \) for \( x \neq 0 \) and \( g(0) = 1 \). The limit of both functions at 0 is 0, and \( f(x) < g(x) \) for all \( x \).
5. Give an example of one of the following strange situations. It doesn’t make any sense to award extra credit for doing both parts, but you can get an extra credit point by explaining why it doesn’t.

(a) Give an example of functions $f$ and $g$ such that $\lim_{x \to 0} f(x)$ and $\lim_{x \to 0} g(x)$ are both undefined, but $\lim_{x \to 0} f(x)g(x)$ exists.

$f(x) = \frac{|x|}{x} = g(x)$ works. The two functions (which are the same of course) have a jump discontinuity at 0 (the function is negative one below 0 and one above). Their product is one everywhere except at 0, and so has limit 1 at 0.

(b) Give an example of a function $f$ which is discontinuous everywhere but has the property that the function $g$ defined by $g(x) = f(x)^2$ is continuous everywhere.

Define $f(x) = 1$ for $x$ rational and -1 for $x$ irrational.

The reason it would not make sense to give extra credit for doing both is that any solution to the second problem is also a solution to the first problem.
2 Additional Problems

These are problems suggested to me by difficulties you had in Homework 4. Some of them might be alternative test questions. There is no guarantee that a question on the test will necessarily look like the ones on the Practice Test (while on the other hand some of the questions on the Practice Test might be on the actual test).

1. We say that a function $g$ is nonegative iff $g(x) \geq 0$ for every $x$ in the domain of $g$. Show that any function $f$ can be expressed as the difference of two nonnegative functions $f_+ - f_-$, where in addition $f_+(x) \cdot f_-(x) = 0$ for every $x$ in the domain of $f$. Give definitions of the functions $f_+$ and $f_-$ in terms of $f$. It is possible to write formulas for $f_+$ and $f_-$ using absolute value, but the most natural way to define them is piecewise. $f_+$ and $f_-$ are uniquely determined by $f$, and they are both straightforward to describe.

   We know that one of the functions must be zero and the other must be nonnegative. So for any $x$ for which $f(x) \geq 0$, the only possibility is that $f_+(x) = f(x)$ and $f_-(x) = 0$. For any $x$ for which $f(x) \leq 0$, the only possibility is that $f_+(x) = 0$ and $f_-(x) = -f(x)$.

   so the definitions are $f_+(x) = f(x)$ for all $x$ for which $f(x) \geq 0$ and $0$ for all $x$ for which $f(x) \leq 0$, while $f_-(x) = -f(x)$ for all $x$ for which $f(x) \leq 0$ and $0$ for all $x$ for which $f(x) \geq 0$.

   If you are cunning you might notice that $f_+(x) = \frac{f(x) + |f(x)|}{2}$ and $f_-(x) = \frac{f(x) - |f(x)|}{2}$.
2. Prove that \( \lim_{x \to 2} \frac{1}{x^2} = \frac{1}{4} \) using an explicit computation of \( \delta \) in terms of \( \epsilon \), and including a proof as well as scratch work.

**Scratch work** We need to make \( \left| \frac{1}{x^2} - \frac{1}{4} \right| < \epsilon \) by controlling \( |x - 2| \).

\[
\left| \frac{1}{x^2} - \frac{1}{4} \right| = \left| \frac{4-x^2}{4x^2} \right| = \frac{|x^2-4|}{4x^2} = \frac{|x-2||x+2|}{4x^2}. \]

We need an upper bound on \( |x + 2| \) and a lower bound on \( 4x^2 \), both in order to bound \( \frac{|x-2||x+2|}{4x^2} \) above. If we require \( |x - 2| < 1 \), we get \( 1 < x < 3 \), which gives us an upper bound of \( 3+2 = 5 \) on \( |x + 2| \) and a lower bound of \( 4(1)^2 \) on \( 4x^2 \), so we have \( \frac{|x-2||x+2|}{4x^2} < \frac{5|x-2|}{4} \) so it is sufficient to make \( |x - 2| < \frac{4}{5} \epsilon \).

**Proof:** Let \( \epsilon > 0 \) be arbitrary.

Let \( x \) be arbitrary.

Assume \( 0 < |x - 2| < \min(1, \frac{4}{5} \epsilon) \).

Note that \( 1 < x < 3 \).

\[
\left| \frac{1}{x^2} - \frac{1}{4} \right| = \frac{|4-x^2|}{4x^2} = \frac{|x^2-4|}{4x^2} = \frac{|x-2||x+2|}{4x^2} < \frac{\frac{5|x-2|}{4}}{4(1)^2} = \frac{5}{4} |x - 2| < \frac{5}{4} \left( \frac{4}{5} \epsilon \right) = \epsilon
\]
3. Chapter 4, problem 2. This is a completely computational question. No vagueness. Give explicit formulas for $t$ in each part. Prove all the inequalities claimed under the given hypotheses, using basic properties of inequalities.

One does need to assume that $a < b$ throughout, so that $b - a > 0$, a fact which is used in several places.

(a) Suppose $x \in [0, b]$. Show that $x = tb$ for some $t$ with $0 \leq t \leq 1$. Give a formula for $t$ and prove that the inequality holds.

**Solution:** $x = tb \rightarrow \frac{x}{b} = t$. $0 \leq x \leq b$ implies $0b^{-1} < xb^{-1} < bb^{-1}$, that is $0 < t < 1$.

(b) Suppose $x \in [a, b]$. Show that there is $t$ with $0 \leq t \leq 1$ such that $x = (1 - t)a + tb$. That is, give a formula for $t$ and then prove explicitly that $0 \leq t \leq 1$.

**Solution:** $x = (1-t)a + tb \rightarrow x = a - ta + tb \rightarrow x - a = t(b - a) \rightarrow t = \frac{x-a}{b-a}$.

$a \leq t \leq b \rightarrow a - a \leq x - a \leq b - a \rightarrow 0 \leq (x-a)(b-a)^{-1} < (b-a)(b-a)^{-1} \rightarrow 0 \leq t \leq 1$.

(c) Prove that if $0 \leq t \leq 1$ then $(1-t)a + tb \in [a, b]$.

$0 \leq t \leq 1 \rightarrow 0 \leq t(b-a) \leq b-a \rightarrow a \leq t(b-a) + a \leq b \rightarrow a \leq (1-t)a + tb < b$

(d) Prove that the numbers in the open interval $(a, b)$ are exactly the numbers $(1-t)a + tb$ with $0 < t < 1$ (this involves implications in both directions).

This is exactly the same reasoning as in parts b and c with $<$ in place of $\leq$ everywhere.
4. Prove that \( \lim_{x \to 1} f(x) = \lim_{x \to 0} f(x + 1) \). Notice that this needs to be argued in both directions: if either limit exists, you need to show that the other exists and is the same.

Suppose (1) \( \lim_{x \to 1} f(x) = L \).

Our goal is to show that \( \lim_{x \to 0} f(x + 1) = L \).

Let \( \epsilon > 0 \). Our goal is to show \( |f(x + 1) - L| < \epsilon \).

Choose \( \delta \) so that for any \( x \), if \( 0 < |x - 1| < \delta \) then \( |f(x) - L| < \epsilon \). We can do this using limit fact (1).

Suppose \( 0 < |x - 0| < \delta \). Then \( 0 < |(x + 1) - 1| < \delta \), so by the previous line with \( x + 1 \) instead of \( x \), we have \( |f(x + 1) - L| < \epsilon \).

Now for the second part of the proof.

Suppose (2) \( \lim_{x \to 0} f(x + 1) = L \). Our goal is to show \( \lim_{x \to 1} f(x) = L \).

Let \( \epsilon > 0 \) be arbitrary.

Choose \( \delta \) so that for any \( x \) if \( 0 < |x - 0| < \delta \) then \( |f(x + 1) - L| < \epsilon \) (using limit fact (2)).

Now suppose that \( 0 < |x - 1| < \delta \) Our goal is to show \( |f(x) - L| < \epsilon \).

We have \( 0 < |(x - 1) - 0| < \delta \) so by two lines above with \( x - 1 \) in place of \( x \) we have \( |f((x - 1) + 1) - L| < \epsilon \), that is, \( |f(x) - L| < \epsilon \).