You have from 12:40 to 1:35 to do the exam. You need nothing but your test paper and your writing instrument. Your paper includes a list of axioms and theorems which you can tear off for convenience. Please carefully note any instructions about what you are allowed to use in any given proof. You may use any theorem you prove on your test paper to support your other proofs (as long as you dont end up arguing in a circle). You are also allowed to prove lemmas of your own in order to prove a theorem.

Test taking strategy: make sure you do problems 1-3 and at least one of 4-6 first. Then keep working and do as much as you can. I’m not saying that I’m dropping anything; a strong performance by most of the class might mean that all problems count equally. But I would like to be free to use a strategy where I count on you to do 1-3 and then use your best work from 4-6 on top of that.
1. A proof is given without any justifications for the lines. Supply them. You will have a use for this result in a later question. Supply the justifications for each line, and replace all the Your Label Here labels with the correct headings.

**Goal:** \((\forall x. x + 0 = 0 + x)\) We prove this by [say what technique]?

**Your Label Here (what is this in the proof?)** \(0 + 0 = 0 + 0\)

1: \(0 + 0 = 0 + 0\)

**Your Label Here (what is this in the proof?)** [it is line 2 as well]

\(k + 0 = 0 + k\)

**Your Label Here (what is this in the proof?):** \(S(k) + 0 = 0 + S(k)\)

3: \(0 + S(k) = S(0 + k)\)

4: \(0 + S(k) = S(k + 0)\)

5: \(k + 0 = k\)

6: \(0 + S(k) = S(k)\)

7: \(S(k) = S(k) + 0\) ax 6

8: \(0 + S(k) = S(k) + 0\)

9: \(S(k) + 0 = 0 + S(k)\)
2. Prove the theorem $(\forall xy. S(x) + y = S(x + y))$. You should not use anything but the axioms.
3. Prove the commutative law of addition \((\forall xy. x + y = y + x)\).
4. Prove the left distributive law of multiplication over addition,

\[(\forall xy. (x + y)z = xz + yz).\]

You may use the commutative and associative properties of addition, and the axioms. You may indicate applications of the commutative and associative laws of addition briefly, but do show each step.
5. Prove the identity property of multiplication, \(1x = x1 = x\). You will need the definition of 1 as \(S(0)\). You may use theorems about addition but not theorems about multiplication (in particular, not commutativity of multiplication). There are two things to prove: one requires induction and one does not.
6. Prove the cancellation property for addition,

\[(\forall x y z. x + z = y + z \rightarrow x = y)\].

I suggest stripping off the first two quantifiers the quick way, followed by induction on \(z\). Axiom 4 comes in handy.
1 Axioms and Theorems of Formal Arithmetic

Here are the axioms of formal arithmetic.

1. 0 is a natural number (in symbols, 0 ∈ N).

2. If x and y are natural numbers, so are S(x), x + y, and x · y. (∀xy ∈ N.S(x) ∈ N ∧ x + y ∈ N ∧ x · y ∈ N).

3. 0 is not a successor. (∀x.S(x) ≠ 0). Here we understand that x ≠ y abbreviates ¬x = y. Here and in the following axioms we write our quantifiers unrestricted: we could write (∀x ∈ N.S(x) ≠ 0) instead, but in this context we are only talking about natural numbers, so we can leave the restriction on our quantifiers implicit.

4. Numbers with the same successor are the same. (∀xy.S(x) = S(y) → x = y).

5. Let P(x) be any sentence about a natural number variable x. We assert P(0) ∧ (∀y.P(y) → P(S(y))) → (∀x.P(x)). This is a symbolic presentation of the familiar principle of mathematical induction. From an extremely technical standpoint, this is an infinite collection of axioms, one for each sentence P(x). If we are also willing to talk about sets of natural numbers, we can state it as a single axiom: (∀A ∈ P(N).0 ∈ A ∧ (∀y ∈ N.y ∈ A → S(y) ∈ A) → A = N). We will not use the set formulation now but we might use it later. P(N) is a notation for the collection of all sets of natural numbers.

6. (∀x.x + 0 = x)

7. (∀xy.x + S(y) = S(x + y))

8. (∀x.x · 0 = 0)

9. (∀x.x · S(y) = x · y + x) Here we assume the usual order of operations.

Here are some theorems and a definition. Not all of these are necessarily of any use.

definition of 1: 1 is defined as S(0).

Theorem 1: (∀x.x + 1 = S(x))
Theorem 2: $\forall x. x = 0 \lor (\exists y. S(y) = x)$

commutativity of addition: $\forall xy. x + y = y + x$

associativity of addition: $\forall xyz. (x + y) + z = x + (y + z)$