Take Home Question on Test II, Math 314

Dr. Holmes

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The content of the question is simple: give a full proof of the result stated in problem 11.6.

I gave an outline of how to prove it in class: what I am testing here is not the clever idea of how to prove the theorem, but the ability to write up the proof in sensible English and with the correct logic.

Please fill in the following outlined proof.

Theorem: If a function f is continuous at a, continuous at b, and increasing on the open interval (a, b), then it is increasing on the closed interval $[a, b]$.

Outline of Proof: Choose arbitrary real numbers $a < b$. Choose an arbitrary function f which is continuous at a, continuous at b and increasing on (a, b) [this last means that for any $x < y$ in (a, b) we have $f(x) < f(y)$].

Our goal is then to prove that f is increasing on $[a, b]$, that is, for any x, y such that $a \leq x < y \leq b$, we have $f(x) < f(y)$.

Choose arbitrary c, d such that $a \leq c < d \leq b$. Our goal is to prove $f(c) < f(d)$.

There are four cases, depending on whether $a = c$ and whether $d = b$. Our goal in each case is to prove $f(c) < f(d)$.

$a < c < d < b$: There is nothing to prove here. That $f(c) < f(d)$ follows from the fact that f is increasing on (a, b).

$a = c < d < b$: This is the case I outlined in class.

Our goal is to prove $f(a) = f(c) < f(d)$. Our strategy is to assume the contrary and reason to a contradiction. So, we assume $f(a) \geq f(d)$ and our goal is now a contradiction.

Choose any e such that $a < e < d$. We have $f(e) < f(d)$, because f is increasing on (a, b), and $f(d) \leq f(a)$ by assumption. Thus we have $f(a) > f(e)$.

We know that f is continuous at a, so for any $\epsilon > 0$ there is a $\delta > 0$ such that for any x, if $|x - a| < \delta$ then $|f(x) - f(a)| < \epsilon$.

In particular, let $\epsilon = f(a) - f(e)$. There is a $\delta > 0$ such that for any x, if $|x - a| < \delta$ then $|f(x) - f(a)| < f(a) - f(e)$. Notice that if any specific positive value for δ works, so does any smaller value, so we can assume $\delta < e - a$.

Prove that $|f(x) - f(a)| < f(a) - f(e)$ implies $f(e) < f(x)$ (fill in details algebraically using properties of inequalities and absolute values: pictorially it is pretty obvious). Explain why this leads to a contradiction.

$a < c < d = b$: Write this out in detail. It is analogous to the previous case.

$a = c < d = b$: Explain how $f(a) < f(b)$ follows from the previous two cases. This is short.