Math 314 Test II Practice Problems

Dr. Holmes

November 16, 2009

Some of these will look very familiar. Some of them probably will appear exactly as you see them on the exam. The point is to understand exactly how to write out the proofs when faced with just that test paper and your writing instrument.

1. Show that the function $f(x)$ defined as x if x is rational and 0 if x is irrational is discontinuous at 1. Show that it is continuous at 0. Your arguments should involve ϵ and δ.

2. Prove that if f is continuous at a then for any $\epsilon > 0$ there is $\delta > 0$ such that if $|x - a| < \delta$ and $|y - a| < \delta$ then $|f(x) - f(y)| < \epsilon$ (chapter 6, problem 15).

3. State the Intermediate Value Theorem (in whatever form you prefer to use) and use it to prove that the equation $x^5 + x + 1 = 0$ has a solution. Be sure to state all conditions needed for the theorem to apply.

4. Use Theorem 1 of chapter 8 (if f is continuous on $[a, b]$, $f(a) < 0$, $f(b) > 0$ then there is c in (a, b) such that $f(c) = 0$) to prove the more general theorem "if f is continuous on $[a, b]$, $f(a) < d < f(b)$, then there is c in $[a, b]$ such that $f(c) = d$". Hint: write a different function to which Theorem 1 applies which can be used to prove this result. Be sure to verify that Theorem 1 applies to your other function and clearly explain why the result of Theorem 1 for the other function implies the desired result here. Similarly, prove that "if f is continuous on $[a, b]$, $f(a) > d > f(b)$, then there is c in $[a, b]$ such that $f(c) = d$". Congratulations: you have proved the usual Intermediate Value Theorem.

5. (a) Prove that the set of natural numbers is not bounded above (i.e., there is no number x such that for all $n \in \mathbb{N}$, $n \leq x$). Hint: if there were an upper bound x for the natural numbers, there would be a *least* upper bound b. $b - 1$ then would not be an upper bound. Argue to a contradiction.

   (b) Prove that for every $x > 0$, there is a natural number n such that $\frac{1}{n} < x$. Hint: use the previous part.

   (c) Prove that for every $x > 0$, there is a natural number n such that $\frac{1}{2^n} < x$.

6. A detail of the bisection method: suppose that we have a sequence of intervals $[a_n, b_n]$ such that for each n, the next interval $[a_{n+1}, b_{n+1}]$ is either the lower half $[a_n, \frac{a_n+b_n}{2}]$ or the upper half $[\frac{a_n+b_n}{2}, b_n]$ of the previous interval. We know by a theorem that we are not asked to prove here that at least one number c belongs to all the intervals $[a_n, b_n]$.

   Show that there is at most one c belonging to all those intervals by the following strategy: assume there are two distinct such numbers c_1 and c_2, then use the last part of the previous problem to derive a contradiction.

7. Suppose A and B are two nonempty sets of numbers such that for all $x \in A$, for all $y \in B$, $x \leq y$. Show that for all $y \in B$, $\sup(A) \leq y$. Then show that $\sup(A) \leq \inf(B)$. This is problem 12, section 8. If your proof is complicated, it is wrong: this is all about understanding the definitions of upper bound, lower bound, least upper bound, greatest lower bound.

8. Define $f(x) = x^2 \sin(\frac{1}{x^2})$ for $x \neq 0$ and define $f(0) = 0$.

   Show that $f'(0) = 0$ (this will involve evaluating a limit, all the way down to epsilons and deltas).

   Show that f' is unbounded on any interval $(-c, c)$. (You could do this in M170).

9. Prove that for any function f and number a such that $f'(a) > 0$, there is a number δ such that for any $x \neq a$ in $(a - \delta, a + \delta)$, $f(x) < f(a)$ iff $x < a$.

   Explain why this does *not* show that f is increasing on $(a - \delta, a + \delta)$.

10. The Critical Point Theorem says that if f is defined on (a, b) and $f'(c)$ is defined for a c in (a, b) such that $f(c)$ is a maximum or minimum value, then $f'(c) = 0$.

    You may assume the Critical Point Theorem and the Extreme Value Theorem. Prove Rolle's Theorem (if f is continuous on $[a, b]$ and differentiable on (a, b) and $f(a) = f(b)$, then there is c in (a, b) such that $f'(c) = 0$) Be sure to state clearly where you used the allowed theorems.

11. In this problem you may use the Mean Value Theorem. It is assumed that you know the statement of this theorem.

    Prove that if f is differentiable on (a, b) and $f'(x) = 0$ for all x in (a, b) then x is constant on (a, b).

    Prove that if f is differentiable on (a, b) and $f'(x) > 0$ for all x in (a, b) then x is increasing on (a, b).

    Prove that if f is differentiable on (a, b) and $f'(x) > M$ for all x in (a, b) then $f(b) > f(a) + M(b - a)$.