Study materials for Test II

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March 10, 2013

This contains some words about problems to read on the old Test II and Test III papers and solutions to Homework 5. I’m busy grading Homework 5, emails should show up sometime tomorrow I hope, but at the latest Tuesday morning.

1 Study guide for the sample tests

All the problems on Test II look like things I could ask. I am disinclined to ask anything like 7, which is entirely about absolute values. In problems like 3 and 4, use only our current axioms (no IA1 IA2 or IA3). I would definitely think about 1b. I would definitely think about 6. All the others except 7 are worth writing out.

In Test III, 1 is not on our test (we haven’t covered the prerequisites). 2 is a prime test question: I would study the proof of the Ray Theorem. 3 is a prime test question; so is the related question in your current homework. 4 is a lovely question. 5 is not covered (later material). 6 is not covered (later material). Working through 7 would be very good for you! I might compose a question like this which requires you to justify steps in a proof which is given.

2 Solutions to Homework 5

As I write this, I have marked problems 3.2.8 and 3.2.11 on all papers. I may add more comments to this document as I mark other questions.

Everything here is fair game for the test. I would definitely be ready for either problem 19 or problem 20 in section 3.2.
Questions about the material in Homework 4 are also fair. Taxicab metric.
Why does a line have an infinite number of points?

3.2.8: Part a.
Let \( P = (x_1, y_1) \) and \( Q = (x_2, y_2) \) throughout.
\[
d(P, Q) = d(Q, P) \text{ because } \max\{ |x_1 - x_2|, |y_1 - y_2| \} = \max\{ |x_2 - x_1|, |y_2 - y_1| \} \text{ by properties of absolute value. I didn’t read your mind here: I wanted you to either say or show me that you were using the property } |a - b| = |b - a|; \text{ if I did not see absolute differences written in both directions I took a little off.}
\]
\[
d(P, Q) \geq 0 \text{ because } \max\{ |x_2 - x_1|, |y_2 - y_1| \}, \text{ the maximum of two nonnegative (not “positive”) numbers is nonnegative (not “positive”). This time I took off a little for confusing these words.}
\]
If \( P = Q \) then \( d(P, Q) = \max\{ |x_1 - x_1|, |y_1 - y_1| \} = 0. \)
If \( d(P, Q) = \max\{ |x_2 - x_1|, |y_2 - y_1| \} = 0, \) then clearly \( |x_2 - x_1| \) and \( |y_2 - y_1| \) are both zero, as the \( \text{max} = 0 \) is greater than or equal to each of the two differences and they are each greater than or equal to zero. So \( x_2 = x_1 \) and \( y_2 = y_1 \) so \( P = Q. \)

The picture in part b should be the square with corners at the four points \((\pm 1, \pm 1)\)

3.2.11: There were two popular solutions.
The simplest is this: if the line has slope less than 1, \( f(x, y) = x. \) If the line has slope greater than 1 or is vertical, \( f(x, y) = y. \)
The most popular one for students to approximate is this: if the line has slope less than 1, \( f(x, y) = x. \) If the line has slope \( m > 1, f(x, y) = f(x, mx + b) = mx \) (notice that this is shifted by a constant from the previous function). If the line is vertical (many students overlooked this) \( f(x, y) = y. \)

3.2.16: I’m not typesetting this with fancy overstrikes: I’ll say things like ray \( AB. \)
Show that if \( C \) is on ray \( AB \) and not equal to \( A \) then ray \( AB \) is the same as ray \( AC. \)
Proof: Assume that \( C \) is on ray \( AB \) and not equal to \( A. \)

2
Let \( f \) be a coordinate function on line \( AB \) such that \( f(A) = 0 \) and \( f(B) > 0 \).

A point \( P \) will be on ray \( AB \) iff \( P = A \) (\( f(A) = 0 \)) \( P = B \) (\( f(A) = f(B) \)) \( A \ast P \ast B \ast f(A) < f(P) < f(B) \) or \( A \ast B \ast P \) (\( f(A) < f(B) < f(P) \)). We do not have to consider the case of the other order because we have fixed the order of \( f(A) \) and \( f(B) \). Notice that the values that \( f(P) \) can take on are just exactly the values \( \geq 0 \).

Because \( C \) is on ray \( AB \) and \( C \) is not \( A \), \( f(C) > 0 \).

But then by exactly the same analysis, the points \( P \) on ray \( AC \) are exactly the points \( P \) such that \( f(P) \geq 0 \).

So ray \( AB \) and ray \( AC \) are the same set of points.

3.2.17: Let \( A, B \) be distinct points. Show that there is exactly one point such that \( A \ast P \ast B \) and \( d(A, P) = d(P, B) \) (there is exactly one midpoint between the two points).

Choose a coordinate function \( f \) on line \( AB \) such that \( f(A) = 0 \) and \( f(B) > f(A) \). \( f(B) = f(B) - 0 = f(B) - f(A) = |f(B) - f(A)| = d(A, B) \).

There is a point \( P \) on line \( AB \) such that \( f(P) = \frac{d(A, B)}{2} \). \( d(A, P) = |\frac{d(A, B)}{2} - 0| = \frac{d(A, B)}{2} \). \( d(P, B) = |f(B) - f(P)| = |d(A, B) - \frac{d(A, B)}{2}| = \frac{d(A, B)}{2} \). So \( P \) is a midpoint.

Suppose that \( Q \) was a midpoint and distinct from \( P \). \( d(A, Q) \) would have to be \( \frac{d(A, B)}{2} \), so \( |f(Q) - f(A)| = |f(Q)| = \frac{d(A, B)}{2} \) so \( f(Q) \) is either \( \frac{d(A, B)}{2} \) or \( -\frac{d(A, B)}{2} \). In the first case \( Q = P \) because \( f \) is a bijection. In the second case, \( d(Q, B) = |f(B) - f(Q)| = |d(A, B) - (-\frac{d(A, B)}{2})| = \frac{3}{2}d(A, B) \), and \( Q \) is not a midpoint. Neither case is possible, so there are no midpoints other than \( P \).

3.2.20: Again, I'm not typesetting this with fancy overstrikes. I'll say line \( AB \), ray \( AB \), segment \( AB \),

Suppose \( A \ast B \ast C \), \( D \ast E \ast F \), segment \( AB \) is congruent to segment \( DE \), and segment \( AC \) is congruent to \( DF \). Show that segment \( BC \) is congruent to segment \( EF \).

Proof: Suppose \( A \ast B \ast C \), \( D \ast E \ast F \), segment \( AB \) is congruent to segment \( DE \), and segment \( AC \) is congruent to \( DF \).
Choose a coordinate function $f$ for line $ABC$ such that $f(A) < f(B) < f(C)$.

Choose a coordinate function $g$ for line $DEF$ such that $g(D) < g(E) < g(F)$.

Segment $AB$ congruent to segment $DE$ means $d(A,B) = d(D,E)$, so $f(B) - f(A) = g(E) - g(D)$ (why don’t I have to write any absolute values?)

Segment $AC$ congruent to segment $DF$ means $d(A,C) = d(D,F)$ so $f(C) - f(A) = g(F) - g(D)$.

Now $d(B,C) =$ [def coordinate function]

\[
\begin{align*}
(f(C) - f(A)) & - (f(B) - f(A)) = [\text{equations above}] \\
(g(F) - g(D)) & - (g(E) - g(D)) = [\text{algebra}] \\
g(F) - g(E) & = [\text{def coordinate function}]d(E, F)
\end{align*}
\]

and $d(B,C) = d(E,F)$ implies by definition that segment $BC$ is congruent to segment $EF$ which is what to be shown.

Either problem 19 or problem 20 could be a test question.

**3.3.1:** Show that the intersection of two convex sets is convex.

Suppose that $C$ is a convex set and $D$ is a convex set.

Our aim is to show that $C \cap D$ is a convex set, that is, that if $A \in C \cap D$ and $B \in C \cap D$ then segment $AB$ is a subset of $C \cap D$. Suppose that $P$ is a point on segment $AB$. Because $C$ is convex and $A, B$ are both in $C$, $P$ is in $C$. Because $D$ is convex and $A, B$ are both in $D$, $P$ is in $D$. Because $P$ is in both $C$ and $D$, $P$ is in $C \cap D$, which completes the proof that segment $AB$ is a subset of $C \cap D$, so $C \cap D$ is convex.

**3.3.3:** Suppose that $L$ is a line and $H$ is one of the half-planes cut by $L$.

Show that $H \cup L$ is convex.

What we need to show is that if $A$ and $B$ are distinct points in $H \cup L$ then segment $AB$ is a subset of $H \cup L$.

There are three cases: either both points are in $L$, both points are in $H$, or one (we can call it $A$ without loss of generality) is on $L$ and one (which we will call $B$) is in $H$. 

4
both points are in $H$: $H$ is convex, so any point on segment $AB$ is in $H$, so any point on segment $AB$ is in $H \cup L$.

both points are in $L$: Any point on segment $AB$ is then in $L$ by definition of a segment, so also in $H \cup L$.

$A$ is on $L$ and $B$ is in $H$: Suppose that a point $C$ on segment $AB$ was not in $H \cup L$ for the sake of a contradiction. Then it must be in the other half-plane cut by $L$. The segment $CB$ must then intersect $L$ by the PSP, since $C$ and $B$ are on different sides of $L$. The line $BC$ equals the line $AB$ which intersects line $L$ at $A$ and so only at $A$ (line $AB$ is different from $L$ because $B$ is not on $L$). But then $A$ is in segment $BC$, so $B \ast A \ast C$, so not $A \ast C \ast B$ (corollary proved in class) so $C$ is not in the segment $AB$, contradiction. So every point in segment $AB$ is in $H \cup L$.

We get the desired conclusion in every case, so we are done.

3.3.6: Suppose that angle $BAC$ and angle $EDF$ exist and are equal. Then $A = D$.

The argument I give here is similar but not identical to the one I gave in class.

Suppose that $A \neq D$ for the sake of a contradiction.

Then $D$ is either in ray $AB$ or ray $AC$. We can suppose it is in ray $AB$: if it were in ray $AC$, the argument would go in exactly the same way with $B$ and $C$ swapped.

We claim that $E$ must also be on ray $AB$.

Suppose otherwise, for the sake of a contradiction. Then $E$ is on ray $AC$, not equal to $A$ (because not on $AB$) and ray $AB$ and ray $AC$ are different.

The ray $DE$ is included in the angle, so we can choose a point $X$ on ray $DE$ different from $D$ or $E$, and it will be on the angle as well.

Suppose $X$ is on $AC$. We then have $E$ and $X$ distinct points on line $AC$ so line $EX$ equals line $AC$.

Also $E$ and $X$ are both on line $DE$ so line $DE$ equals line $AC$.

Now $A$ and $D$ are both on this line and $A$ and $D$ are also both on line $AB$. So line $AB$ equals line $AC$. From this, since ray $AB$ and ray
$AC$ are distinct, we conclude $B \ast A \ast C$ so angle $BAC$ does not exist, contradiction.

So $D$ and $E$ are both on ray $AB$.

By exactly the same argument, $D$ and $F$ are both on ray $AB$.

Now $F \ast D \ast E$ is impossible because angle $EDF$ exists, so we have either

$A \ast D \ast E \ast F$ or $A \ast D \ast F \ast E$ – in either of these cases, $A$ is neither in ray $DE$ nor ray $DF$, contradiction

or $A \ast E \ast F \ast D$ or $A \ast F \ast E \ast D$: in either of these cases, we can choose a point $Y$ so that $A \ast D \ast Y$,

so either $A \ast E \ast F \ast D \ast Y$ or $A \ast F \ast E \ast D \ast Y$ and $Y$ is on ray $AD$

which is the same as ray $AB$ so it is in the angle

but also it is on neither ray $DE$ nor ray $DF$, contradiction.