Math 311 Test III, Spring 2012

Dr. Holmes

April 23, 2012

The exam will last from 8:15 am to 9:35 am. You are allowed your writing instrument and your test paper. The instructor will have scratch paper available on request. There is a sheet of axioms at the end of the exam; you are welcome to tear it off and you do not need to return it.

You may drop one complete numbered question. If you do all numbered questions, your best work will count.
1. Care with definitions. We say that $\overrightarrow{AD}$ is a bisector of angle $\angle BAC$ iff $D$ is in the interior of $\angle BAC$ and the measures of $\angle DAC$ and $\angle DAB$ are the same. Why do we need to say that $D$ is in the interior of $\angle BAC$? Draw a picture of a situation where the measures of $\angle DAC$ and $\angle DAB$ are the same and we do not want to say that $\overrightarrow{AD}$ is the bisector of angle $\angle BAC$. 
2. Prove the Ray Theorem: if $L$ is a line and $A$ is incident on that line, and $B$ is an exterior point to $L$, then every point $C \neq A$ on $\overrightarrow{AB}$ is on the same side of $L$ as $B$ (that is, it is in the same one of the two half-planes determined by $L$ that $B$ is in). This is all about betweenness: a proof that does not talk about betweenness relations between points and use the definition of a ray in terms of betweenness is not going to be acceptable. You can use common sense to reason about betweenness relations, but you have to talk about betweenness.
3. Prove Pasch’s Axiom: if \( \triangle ABC \) is a triangle and line \( L \) is not incident on any of the vertices \( A, B, C \), and \( L \) is incident on \( \overline{AB} \), then either \( L \) intersects \( \overline{AC} \) or \( L \) intersects \( \overline{BC} \).
4. Prove that there are three non-collinear points. (The Plane Separation Postulate is useful). In this proof, you must cite each and every axiom that you use and make it reasonably clear how you use it.
5. State and prove the Vertical Angle Theorem (it helps to draw a diagram first). You only need to consider one pair of angles. You may use the Linear Pair Theorem: be sure to identify the linear pairs in your proof.
6. Part of the proof of the angle bisector theorem: suppose that $\angle BAC$ is an angle with measure $\alpha$, and $\overrightarrow{AD}$ is between $\overrightarrow{AB}$ and $\overrightarrow{AC}$ (that is, $D$ is in the interior of $\angle BAC$), and the measures of angle $\angle DAB$ and angle $\angle DAC$ are equal, then $\overrightarrow{AD}$ is the only ray with this property. Hint: this is an argument using different parts of the Protractor Postulate and algebra.
7. Fill in the gaps.

**Lemma**: If $A$, $B$, $C$, $D$ are four distinct points such that $C$ and $D$ are on the same side of line $AB$, and $D$ is not on $\overrightarrow{AC}$, then either $C$ is in the interior of $\angle BAD$ or $D$ is in the interior of $\angle BAC$.

**Proof**: Each lettered part ends where I ask you to do something.

(a) Assume that $C$ and $D$ are on the same side of $\overrightarrow{AB}$ and that $D$ is not on $\overrightarrow{AC}$. This sets us up to prove the implication in the theorem.

Assume that $D$ is not in the interior of $\angle BAC$. Our goal is to prove that $C$ is in the interior of $\angle BAD$.

This is a standard strategy for proving/using (circle one of these verbs) a statement of what logical form?

(b) We know that $C$ and $D$ are on the same side of line $AB$ (hypothesis) and $D$ is not in the interior of $\angle BAC$, so $B$ and $D$ must lie on opposite sides of line $AC$.

Explain why the italicized sentence follows. Hint: you will use the definition of interior of an angle.

(c) Thus $\overrightarrow{BD}$ intersects line $AC$. Why? (this is a one-liner)

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(d) Let $C'$ be the unique point at which $\overline{BD}$ intersects line $AC$. By Theorem 3.3.10, $C'$ is in the interior of $\angle BAD$. In particular, $D$ and $C'$ lie on the same side of line $AB$. Explain briefly why the italicized sentence follows.

(e) Since $C$ and $C'$ both lie on the same side of line $AB$ as $D$, $A$ cannot lie between $C$ and $C'$. Explain. What would go wrong if $C \neq A \neq C'$?

(f) So $\overrightarrow{AC}$ and $\overrightarrow{AC'}$ cannot be opposite rays. So $\overrightarrow{AC} = \overrightarrow{AC'}$. It follows that $C$ is in the interior of $\angle BAD$. Explain. Hint: this conclusion falls apart into two statements using the definition of angle interior. Both of these follow directly from a theorem you have seen recently and the fact that $C'$ is in the interior of $\angle BAD$, noted above in the proof.