# Manual of Logical Style

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1 Introduction

I’m revising this document for Math 287, Spring 2014. Students please note that while the symbolism is useful, it is mostly to be considered as shorthand to help us clarify the meaning of bits of mathematical English. The business of proof is conducted in technical English, in practice, though it can be conducted in formal symbolism, and the rules of reasoning can perhaps be presented more clearly in formal symbolism, though they are basically the same as the rules we use when reasoning in technical English, and the purpose is to learn how to read reasoning in technical English accurately and write reasoning in technical English clearly.

This document is designed to assist students in planning proofs. I will try to make it as nontechnical as I can.

There are two roles that statements can have in a proof: a statement can be a claim or goal, something that we are trying to prove; a statement can be something that we have proved or which we have shown to follow from
current assumptions, that is, a statement which we can use in the current argument. It is very important not to confuse statements in these two roles: this can lead to the fallacy of assuming what you are trying to prove (which is well-known) or to the converse problem, which I have encountered now and then, of students trying to prove things that they already know or are entitled to assume!

In the system of reasoning I present here, we classify statements by their top-level logical operation: for each statement with a particular top-level operation, there will be a rule or rules to handle goals or claims of that form, and a rule or rules to handle using statements of that form which we have proved or are entitled to assume.

In what follows, I make a lot of use of statements like "you are entitled to assume A". Notice that if you can flat-out prove A you are entitled to assume A. The reason I often talk about being entitled to assume A rather than having proved A is that one is often proving things using assumptions which are made for the sake of argument.

2 Conjunction

In this section we give rules for handling “and”. These are so simple that we barely notice that they exist! The symbol we use for the “and” in “A and B” (when A and B are sentences) is “A ∧ B”. “A&B” might be used instead (for example, if one were restricted to a typewriter symbol set).

2.1 Saying a conjunction

Generally conjunctions in English are easy to recognize: “A and B” or “both A and B” (where A and B are sentences). The second form is used to help with grouping, in situations where mathematical notation uses parentheses.

A feature of English that is not present in formalized languages in practice (though it could be added, with care) is the ability to connect noun phrases with and instead of sentences. Here the danger is that not all uses of and between noun phrases are really instances of logical conjunction.

“John and Mary love ice cream” is equivalent to (John loves ice cream) and (Mary loves ice cream) and this is equivalent to a partially symbolic form (John loves ice cream) ∧ (Mary loves ice cream).
But “John and Mary carried the fallen log” probably does not mean (John carried the fallen log) and (Mary carried the fallen log). Common sense suggests that this is something they did together.

Things like this do happen in mathematics. The sentence “x and y are relatively prime” does not mean “(x is relatively prime) and (y is relatively prime)” – in fact, this doesn’t even make sense. But “2 and 7 are odd numbers” does mean “(2 is an odd number) and (7 is an odd number)”. Understanding what is happening in a particular mathematical sentence involves understanding mathematical terminology and usage, as in any technical area.

One thing to watch out for is usages regarding sets. The set consisting of the elements of A and the elements of B, the union of A and B, has as its elements the objects x which either belong to A or belong to B; the set of things which belong to A and belong to B is the intersection of A and B. The boldface use of and in this paragraph is not an occurrence of the logical operator of conjunction.

2.2 Proving a conjunction
To prove a statement of the form $A \land B$, first prove $A$, then prove $B$.

This strategy can actually be presented as a rule of inference:

\[
\begin{align*}
A \\
B \\
\hline
A \land B
\end{align*}
\]

If we have hypotheses A and B, we can draw the conclusion $A \land B$: so a strategy for proving $A \land B$ is to first prove $A$ then prove $B$. This gives a proof in two parts, but notice that there are no assumptions being introduced in the two parts: they are not separate cases.

If we give this rule a name at all, we call it “conjunction introduction”.

2.3 Using a conjunction
If we are entitled to assume $A \land B$, we are further entitled to assume $A$ and $B$. This can be summarized in two rules of inference:

\[
\begin{align*}
A \land B \\
\hline
A
\end{align*}
\]
2.4 Expressing these rules in writing

The conjunction rules are so “simple” that their use is often completely invisible. When we have assumed or proved $A \land B$, we just proceed as if we have assumed or proved each of $A$ and $B$.

3 Implication

In this section we give rules for implication. There is a single basic rule for implication in each subsection, and then some derived rules which also involve negation, based on the equivalence of an implication with its contrapositive. These are called derived rules because they can actually be justified in terms of the basic rules. We like the derived rules, though, because they allow us to write proofs more compactly. We use the symbolic form $A \rightarrow B$ for sentences like “If $A$ then $B$.”

3.1 Saying an implication

Implications (also called conditionals) are everywhere in mathematics, and there are quite a number of different ways that they are expressed.

“If $A$, then $B$”
“$B$ if $A$”
“That $A$ implies that $B$”
“That $A$ is true is sufficient for $B$ to be true”
“That $B$ is true is necessary for $A$ to hold”
and a profusion of variations of these and other forms express implications.

A universal statement like “All men are mortal” in fact expresses an implication “For any $x$, if $x$ is a man then $x$ is mortal”, or just “If $x$ is a man then $x$ is mortal” with the complete generality of $x$ understood. The
element of generality which is also present we will talk about later. Such statements are found in mathematics. “All odd primes are greater than 2”, “All quadrilaterals with diagonals of equal length are rectangles” are examples. It is a basic skill in proof reading and writing to know how to unpack these statements into (general = universally quantified) conditional statements.

The meaning of an implication or conditional is more precise than in ordinary English, and the exact meaning may be surprising to you (though you have encountered it already in Math 187): “If $A$ then $B$” is understood to be false exactly if $A$ is true and $B$ is false.

This has unexpected consequences.

“If Napoleon conquered China, then $2+2=5$” is true. “If Napoleon conquered China, then $2+2=4$” is true. A false statement implies anything.

An example of this which I enjoy is “All wombats in this room are named Alice”. This expands to “If $x$ is a wombat in this room, $x$ is named Alice”. No matter what $x$ is, the hypothesis is false. So the statement is true.

Counterfactual uses of implications in ordinary English don’t translate well.

“If Lee won at Gettysburg, the South would have won the Civil War” is a premise that a historian might write a lengthy paper about. He would not mean the same thing as a mathematician would mean by this: the mathematical misreading of this statement is simply trivially true, because Lee did not win at Gettysburg. The “would have” is of course a signal that the mathematical meaning is not intended. This can be avoided by having Lee himself say on the morning of the battle: “If we win today, the South wins the war”. This is historically debateable, but for the mathematician simply true, because he loses.

“If Lee won at Gettysburg, Caesar would have conquered Germany” is just as true under the mathematical interpretation of conditionals, and absurd for the historian, and makes it really clear what the problem is here: any element of causality in implication is to be disregarded. We are used to a conditional requiring a connection (usually causal) between the two sentences: this is not the case in mathematics.

A context where the mathematical meaning does work well: a mother addresses her child with “If you clean your room, you will have pizza”. If the child cleans his room and gets pizza, the mother was honest. If the child cleans his room and does not get pizza, the mother is a fink. If the child does not clean his room and does not get pizza, this is only to be expected.
If the child does not clean his room and gets pizza anyway, the mother is a chump. But she didn’t break her promise exactly.

The point here is not that the mathematician believes that this is the real meaning of if/then statements. The mathematician is using language in a technical way. The meaning of if/then statements in unusual situations is often quite unclear in informal English; the mathematical definition is precise, and works for our purposes. Under disjunction below we will cite an example where members of another profession (lawyers) introduce a logical connective with a precise and nonstandard meaning (and/or) in order to combat vague usages in ordinary language.

3.2 Proving an implication

The basic strategy for proving an implication: To prove $A \rightarrow B$, add $A$ to your list of assumptions and prove $B$; if you can do this, $A \rightarrow B$ follows without the additional assumption.

Stylistically, we indent the part of the proof consisting of statements depending on the additional assumption $A$: once we are done proving $B$ under the assumption and thus proving $A \rightarrow B$ without the assumption, we discard the assumption and thus no longer regard the indented group of lines as proved.

This rule is called “deduction”

The indirect strategy for proving an implication: To prove $A \rightarrow B$, add $\neg B$ as a new assumption and prove $\neg A$: if you can do this, $A \rightarrow B$ follows without the additional assumption. Notice that this amounts to proving $\neg B \rightarrow \neg A$ using the basic strategy, which is why it works.

This rule is called “(deduction of) contrapositive”

3.3 Using an implication

modus ponens: If you are entitled to assume $A$ and you are entitled to assume $A \rightarrow B$, then you are also entitled to assume $B$. This can be written as a rule of inference:
\[
\begin{align*}
A \\
A \rightarrow B \\
\hline
B
\end{align*}
\]

**when you just have an implication:** If you are entitled to assume \( A \rightarrow B \), you may at any time adopt \( A \) as a new goal, for the sake of proving \( B \), and as soon as you have proved it, you also are entitled to assume \( B \). Notice that no assumptions are introduced by this strategy. This proof strategy is just a restatement of the rule of *modus ponens* which can be used to suggest the way to proceed when we have an implication without its hypothesis.

**modus tollens:** If you are entitled to assume \( \neg B \) and you are entitled to assume \( A \rightarrow B \), then you are also entitled to assume \( \neg A \). This can be written as a rule of inference:

\[
\begin{align*}
A \rightarrow B \\
\neg B \\
\hline
\neg A
\end{align*}
\]

Notice that if we replace \( A \rightarrow B \) with the equivalent contrapositive \( \neg B \rightarrow \neg A \), then this becomes an example of modus ponens. This is why it works.

**when you just have an implication:** If you are entitled to assume \( A \rightarrow B \), you may at any time adopt \( \neg B \) as a new goal, for the sake of proving \( \neg A \), and as soon as you have proved it, you also are entitled to assume \( \neg A \). Notice that no assumptions are introduced by this strategy. This proof strategy is just a restatement of the rule of *modus tollens* which can be used to suggest the way to proceed when we have an implication without its hypothesis.
3.4 Expressing these rules in writing

My usual style manual sheet on this looks like this. My general view about these abstract formats is that their use becomes much clearer when you have seen and constructed some examples. Some instructors will prefer that you not write goal statements, fearing that you might use them: I believe that it is important to write goals to guide what you are doing, but also that it is very important that goals be clearly distinguished from what you are entitled to assume (what you are entitled to assume is either proved, assumed, or derived from assumptions).

Direct proof of an implication:

To prove \( A \rightarrow B \):

Assume: \( A \)

Goal: \( B \)

Proof of \( B \) using assumption \( A \): some mathematical argument, ending with “thus \( B \)”.

Thus \( A \rightarrow B \) has been shown. since \( B \) follows if \( A \) is assumed (this closing bit is optional but might be useful if the proof part is long and complicated)

Indirect (contrapositive) proof of an implication: The indirect approach can be implemented similarly.

To prove \( A \rightarrow B \) (by contrapositive):

Assume: \( \neg B \)

Goal: \( \neg A \)

Proof of \( \neg A \) using assumption \( \neg B \): some mathematical argument ending with “thus \( \neg A \)”.

Thus \( A \rightarrow B \) has been shown. since \( \neg A \) follows if \( \neg B \) is assumed (this closing bit is optional but might be useful if the proof part is long and complicated)

Use of an implication in modus ponens: Modus ponens in mathematical text mostly requires attention to see how the conclusion is being drawn.
Lemma 1: \( A \rightarrow B \)

::

stuff .

::

Lemma 2: \( A \)

Thus \( B \) follows by modus ponens from Lemma 1 and Lemma 2.

**Use of an implication in modus tollens:** or for an application of modus tollens

Lemma 1: \( A \rightarrow B \)

::

stuff .

::

Lemma 2: \( \neg B \)

Thus \( \neg A \) follows by modus tollens from Lemma 1 and Lemma 2.

**Comments on using implications:** But of course either the condition or its hypothesis might be something you are supposed to be currently aware of and might not be an explicitly named lemma or other kind of result. The conclusion and the hypothesis might appear in the other order in either rule.

**Use of an implication by itself:** The strategy where you start with just an implication can be outlined thus:

**Lemma:** \( A \rightarrow B \)

**Goal:** \( A \) (in order to be able to prove \( B \))

**Proof of \( A \):** some mathematical argument, ending with “thus \( A \)”.

**Thus** we can further conclude \( B \) by modus ponens.

You should be able to fill in what a similar argument using modus tollens would look like.
4 Absurdity

The symbol $\bot$ represents a convenient fixed false statement. The point of having this symbol is that it makes the rules for negation much cleaner.

4.1 Saying the absurd

The symbol $\bot$ is a technical device. Its role in an argument in technical English will be played by a contradiction “$A$ and $\neg A$” or some other explicit statement which is known to be false.

4.2 Proving the absurd

We certainly hope we never do this except under assumptions! If we are entitled to assume $A$ and we are entitled to assume $\neg A$, then we are entitled to assume $\bot$. Oops! This rule is called contradiction.

\[
\begin{array}{c}
A \\
\neg A \\
\bot 
\end{array}
\]

This is simply a way to introduce the absurd in the usual situation where we throw up our hands in despair and say that we have arrived at absurdity, in the form of an explicit contradiction.

4.3 Using the absurd

We hope we never really get to use it, but it is very useful. If we are entitled to assume $\bot$, we are further entitled to assume $A$ (no matter what $A$ is). From a false statement, anything follows. We can see that this is valid by considering the truth table for implication.

This rule is called “absurdity elimination”.

The reader should recognize this as an application of the fact that any conditional with a false hypothesis is true, together with modus ponens. If we assume $\bot$ (a false statement), we further know that $\bot \rightarrow A$ is true (vacuous implication) and by modus ponens can conclude $A$. 
5 Negation

The rules involving just negation are stated here. We have already seen derived rules of implication using negation, and we will see derived rules of disjunction using negation below.

5.1 Saying a negation

The form we use symbolically for denying a sentence $A$ is $\neg A$.

The odd thing is that this is not the way we negate statements naturally in English. We can say “It is not the case that $A$”, where $A$ is a sentence, but we actually say “Roses are not red” instead of “It is not the case that roses are red”.

Further, it is unusual to negate logically complex statements in English. Instead of saying “It is not the case that both $A$ and $B$”, we say $\neg A$ or $\neg B$, where of course $\neg A$ and $\neg B$ are also subject to the kinds of transformations we are discussing. “It is not the case both that roses are red and violets are blue” is much less likely an utterance than “Roses are not red or violets are not blue”, in which no sentence is negated at all, just verbs.

Similarly, “It is not the case that either $A$ or $B$” is much more likely to be expressed as “$\neg A$ and $\neg B$”. “It is not the case either that roses are red and violets are blue” is more likely to be expressed as “Roses are not red and violets are not blue”, though with a nod to the special form “Neither are roses red nor are violets blue”.

These are examples of de Morgan’s laws, which state the logical equivalence of $\neg(A \land B)$ with $\neg A \lor \neg B$ and the logical equivalence of $\neg(A \lor B)$ with $\neg A \land \neg B$. Notice that we are not assuming these as basic logical rules (we can prove them from the rules we present): their importance at this point has more to do with understanding the logical transformations which happen when we negate sentences, in mathematical English much as in natural speech.

General statements and existential statements undergo similar transformations.

“It is not the case that for all $x$, $P(x)$” becomes “There is an $x$ such that $\neg P(x)$” (where of course $\neg P(x)$ is likely to be further transformed). This asserts the existence of a counterexample. We can also say things like “Not all roses are red” instead of “Some rose is not red” in place of “It is not the case that all roses are red”.

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“It is not the case that there is an $x$ such that $P(x)$” becomes the equivalent “For all $x$, $\neg P(x)$” (though there is also the form “For no $x$, $P(x)$”.

“It is not the case that all men are mortal” becomes “Some men are not mortal”.

“It is not the case that some even primes are greater than 2” becomes “All even primes are $\leq 2$”.

(This is a section where I want to think about adding more examples).

There are a lot of ways that negative statements can be transformed: the general rule is that explicitly saying that a sentence is false is not a natural thing to do in English, mathematical or otherwise, though it may be done for emphasis.

5.2 Proving a negation

direct proof of a negation (basic): To prove $\neg A$, add $A$ as an assumption and prove $\bot$. If you complete this proof of $\bot$ with the additional assumption, you are entitled to conclude $\neg A$ without the additional assumption (which of course you now want to drop like a hot potato!). This is the direct proof of a negative statement: proof by contradiction, which we describe next, is subtly different.

Call this rule “negation introduction”.

proof by contradiction (derived): To prove a statement $A$ of any logical form at all, assume $\neg A$ and prove $\bot$. If you can prove this under the additional assumption, then you can conclude $A$ under no additional assumptions. Notice that the proof by contradiction of $A$ is a direct proof of the statement $\neg \neg A$, which we know is logically equivalent to $A$; this is why this strategy works.

Call this rule “reductio ad absurdum”.

5.3 Using a negation:

double negation (basic): If you are entitled to assume $\neg \neg A$, you are entitled to assume $A$.

contradiction (basic): This is the same as the rule of contradiction stated above under proving the absurd: if you are entitled to assume $A$ and
you are entitled to assume \( \neg A \), you are also entitled to assume \( \bot \). You also feel deeply queasy.

\[
\begin{array}{c}
A \\
\neg A \\
\bot
\end{array}
\]

**if you have just a negation:** If you are entitled to assume \( \neg A \), consider adopting \( A \) as a new goal: the point of this is that from \( \neg A \) and \( A \) you would then be able to deduce \( \bot \) from which you could further deduce whatever goal \( C \) you are currently working on. This is especially appealing as soon as the current goal to be proved becomes \( \bot \), as the rule of contradiction is the only way there is to prove \( \bot \).

### 5.4 Style manual formats for negation

#### Direct proof of a negation:

**To prove \( \neg A \):**

**Assume:** \( A \) (for the sake of a contradiction)

**Goal:** A contradiction or absurd statement. If you know what absurdity you intend to prove, stating it as a goal makes sense.

**Proof of an absurd statement:** A body of mathematical argument ending with “thus \( X \), which is absurd”.

**Thus:** \( \neg A \), since the assumption of \( A \) leads to absurdity (this is optional, but may be advisable to make it clear what has been done).

**Proof by contradiction:** This is very similar to the preceding but not the same. A statement of any logical form can be proved by contradiction: the proof by contradiction of \( A \) is the direct proof of the negation \( \neg \neg A \).

**To prove \( A \):**

**Assume:** \( \neg A \) (for the sake of a contradiction) [It is very likely that this statement will be transformed as we describe above under “saying a negation” before being used.]

**Goal:** A contradiction or absurd statement. If you know what absurdity you intend to prove, stating it as a goal makes sense.
Proof of an absurd statement: A body of mathematical argument ending with “thus X, which is absurd”.

Thus: A, since the assumption of ¬A leads to absurdity (this is optional, but may be advisable to make it clear what has been done).

6 Disjunction

In this section, we give basic rules for disjunctions (sentences A or B, written \( A \lor B \)) which do not involve negation, and derived rules which do. The derived rules can be said to be the default strategies for proving a disjunction, but they can be justified using the seemingly very weak basic rules (which are also very important rules, but often used in a “forward” way as rules of inference).

The basic strategy for using an implication (proof by cases) is of course very often used and very important. The derived rules in this section are justified by the logical equivalence of \( P \lor Q \) with both \( \neg P \rightarrow Q \) and \( \neg Q \rightarrow P \): if they look to you like rules of implication, that is because somewhere underneath they are.

6.1 Saying a disjunction

The same issues exist about linking noun phrases with “or” that exist with linking noun phrases with “and”. “My coat or my sweater will keep me warm” means \((\text{My coat will keep me warm}) \lor (\text{my sweater will keep me warm})\).

There is a dichotomy between senses of the word “or” which creates ambiguities in English. Sometimes “A or B” is taken to mean that at least one of the two sentences is true (and both might be) [inclusive or] and sometimes it is taken to mean that exactly one of the sentences is true (and they are not both true) [exclusive or]. Lawyers use the artificial word and/or to signal use of the inclusive sense of or. Mathematicians stipulate that the English word “or” and the symbol \( \lor \) always carry the inclusive sense. For a mathematician, \( A \lor B \) means, “either A is true, or B is true, or both.”

There is a much less used notion of exclusive or, which is often given the notation \( A \oplus B \), meaning A is true or B is true but not both. But the English word “or” always has the inclusive meaning in mathematical text.
6.2 Proving a disjunction

the basic rule for proving a disjunction (two forms): To prove $A \lor B$, prove $A$. Alternatively, to prove $A \lor B$, prove $B$. You do not need to prove both (you should not expect to be able to!)

This can also be presented as a rule of inference, called addition, which comes in two different versions.

\[
\begin{array}{c}
A \\
\hline
A \lor B
\end{array}
\]

\[
\begin{array}{c}
B \\
\hline
A \lor B
\end{array}
\]

the default rule for proving a disjunction (derived, two forms): To prove $A \lor B$, assume $\neg B$ and attempt to prove $A$. If $A$ follows with the additional assumption, $A \lor B$ follows without it.

Alternatively (do not do both!): To prove $A \lor B$, assume $\neg A$ and attempt to prove $B$. If $B$ follows with the additional assumption, $A \lor B$ follows without it.

Notice that the proofs obtained by these two methods are proofs of $\neg B \rightarrow A$ and $\neg A \rightarrow B$ respectively, and both of these are logically equivalent to $A \lor B$. This is why the rule works. Showing that this rule can be derived from the basic rules for disjunction is moderately hard.

Call both of these rules “disjunction introduction”.

6.3 Using a disjunction

proof by cases (basic): If you are entitled to assume $A \lor B$ and you are trying to prove $C$, first assume $A$ and prove $C$ (case 1); then assume $B$ and attempt to prove $C$ (case 2).

Notice that the two parts are proofs of $A \rightarrow C$ and $B \rightarrow C$, and notice that $(A \rightarrow C) \land (B \rightarrow C)$ is logically equivalent to $(A \lor B) \rightarrow C$ (this can be verified using a truth table).

This strategy is very important in practice.
**disjunctive syllogism (derived, various forms):** If you are entitled to assume \( A \lor B \) and you are also entitled to assume \( \neg B \), you are further entitled to assume \( A \). Notice that replacing \( A \lor B \) with the equivalent \( \neg B \rightarrow A \) turns this into an example of modus ponens.

If you are entitled to assume \( A \lor B \) and you are also entitled to assume \( \neg A \), you are further entitled to assume \( B \). Notice that replacing \( A \lor B \) with the equivalent \( \neg A \rightarrow B \) turns this into an example of modus ponens.

Combining this with double negation gives further forms: from \( B \) and \( A \lor \neg B \) deduce \( A \), for example.

Disjunctive syllogism in rule format:

\[
\begin{array}{ccc}
A \lor B \\
\neg B \\
\hline
A \\
\end{array}
\]

\[
\begin{array}{ccc}
A \lor B \\
\neg A \\
\hline
B \\
\end{array}
\]

### 6.4 Style manual formats for disjunction

**Rules not discussed:** The rules of addition and disjunctive syllogism seem not to require much discussion. The application of a rule of addition will be very obvious, while the application of disjunctive syllogism will look much like the application of modus ponens or modus tollens (the issues have to do with backward reference as one or both of the two statements used may be far away in the text from where the conclusion is drawn).
Derived rules for proving a disjunction: Since these rules are actually proofs of implications in disguise, they have quite similar formats. There are two forms. Only one of these needs to be used. It has been a common error among students in learning this format for proof of disjunctions to give proofs in both of these styles, which is redundant.

To prove $A \lor B$ (style 1):
- **Assume:** $\neg A$
- **Goal:** $B$
- **Proof of $B$ using assumption $\neg A$:** some mathematical argument, ending with “thus $B$”.

Thus $A \lor B$ has been shown. since $B$ follows if $A$ is assumed to be false. (this closing bit is optional but might be useful if the proof part is long and complicated)

To prove $A \lor B$ (style 2):
- **Assume:** $\neg B$
- **Goal:** $A$
- **Proof of $A$ using assumption $\neg B$:** some mathematical argument, ending with “thus $A$”.

Thus $A \lor B$ has been shown. since $A$ follows if $B$ is assumed to be false. (this closing bit is optional but might be useful if the proof part is long and complicated)
Proof by cases: This very important rule for using (not proving!) a disjunction frequently presents as difficult.

Lemma: $A \lor B$

\[ \text{stuff} \]

\[ \text{Goal: } C \text{ (the form of the goal is irrelevant here), to be proved by cases using the Lemma.} \]

Case 1: Assume: $A$

Goal: $C$

Proof of $C$: Some mathematical argument ending in “thus $C$”

Case 2: Assume: $B$

Goal: $C$

Proof of $C$: Some mathematical argument ending in “thus $C$”

Thus $C$, since it follows in each of the cases permitted by the Lemma.

7 Biconditional

Some of the rules for the biconditional are derived from the definition of $A \leftrightarrow B$ as $(A \rightarrow B) \land (B \rightarrow A)$. There is a further very powerful rule allowing us to use biconditionals to justify replacements of one expression by another.

7.1 Proving biconditionals

the basic strategy for proving a biconditional: To prove $A \leftrightarrow B$, first assume $A$ and prove $B$; then (finished with the first assumption) assume $B$ and prove $A$. Notice that the first part is a proof of $A \rightarrow B$ and the second part is a proof of $B \rightarrow A$.

Call this rule “biconditional deduction”.

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**derived forms:** Replace one or both of the component proofs of implications with the contrapositive forms. For example, one could first assume $A$ and prove $B$, then assume $\neg A$ and prove $\neg B$ (changing part 2 to the contrapositive form).

### 7.2 Using biconditionals

The rules are all variations of modus ponens and modus tollens. Call them biconditional modus ponens or biconditional modus tollens as appropriate.

If you are entitled to assume $A$ and $A \leftrightarrow B$, you are entitled to assume $B$.

If you are entitled to assume $B$ and $A \leftrightarrow B$, you are entitled to assume $A$.

If you are entitled to assume $\neg A$ and $A \leftrightarrow B$, you are entitled to assume $\neg B$.

If you are entitled to assume $\neg B$ and $A \leftrightarrow B$, you are entitled to assume $\neg A$.

These all follow quite directly using modus ponens and modus tollens and one of these rules:

- If you are entitled to assume $A \leftrightarrow B$, you are entitled to assume $A \rightarrow B$.
- If you are entitled to assume $A \leftrightarrow B$, you are entitled to assume $B \rightarrow A$.

The validity of these rules is evident from the definition of a biconditional as a conjunction.

### 7.3 Calculating with biconditionals

Let $F$ be a complex expression including a propositional letter $P$. For any complex expression $C$ let $F[C/P]$ denote the result of replacing all occurrences of $P$ by $C$.

The replacement rule for biconditionals says that if you are entitled to assume $A \leftrightarrow B$ and also entitled to assume $F[A/P]$, then you are entitled to assume $F[B/P]$. Also, if you are entitled to assume $A \leftrightarrow B$ and also entitled to assume $F[B/P]$, then you are entitled to assume $F[A/P]$.

The underlying idea which we here state very carefully is that $A \leftrightarrow B$ justifies substitutions of $A$ for $B$ and of $B$ for $A$ in complex expressions. This is justified by the fact that all our operations on statements depend only on their truth value, and $A \leftrightarrow B$ is equivalent to the assertion that $A$ and $B$ have the same truth value.
This rule and a list of biconditionals which are tautologies motivates the "boolean algebra" approach to logic.

7.4 Expressing this in writing

A proof of a biconditional is always in two parts, the proof of conditional sentences $A \rightarrow B$ and $B \rightarrow A$ (each proved either directly or indirectly, independently of one another): the style manual for the parts is already given.

8 Universal Quantifier

This section presents rules for $(\forall x. P(x))$ ("for all $x$, $P(x)$") and for the restricted form $(\forall x \in A. P(x))$ ("for all $x$ in the set $A$, $P(x)$"). Notice that $(\forall x \in A. P(x))$ has just the rules one would expect from its logical equivalence to $(\forall x. x \in A \rightarrow P(x))$.

8.1 Proving Universally Quantified Statements

To prove $(\forall x. P(x))$, first introduce a name $a$ for a completely arbitrary object. This is signalled by a line "Let $a$ be chosen arbitrarily". This name should not appear in any earlier lines of the proof that one is allowed to use. The goal is then to prove $P(a)$. Once the proof of $P(a)$ is complete, one has proved $(\forall x. P(x))$ and should regard the block beginning with the introduction of the arbitrary name $a$ as closed off (as if "Let $a$ be arbitrary" were an assumption). The reason for this is stylistic: one should free up the use of the name $a$ for other similar purposes later in the proof.

To prove $(\forall x \in A. P(x))$, assume $a \in A$ (where $a$ is a name which does not appear earlier in the proof in any line one is allowed to use): in the context of this kind of proof it is appropriate to say "Let $a \in A$ be chosen arbitrarily" (and supply a line number so the assumption $a \in A$ can be used). One's goal is then to prove $P(a)$. Once the goal is achieved, one is entitled to assume $(\forall x \in A. P(x))$ and should not make further use of the lines that depend on the assumption $a \in A$. It is much more obvious in the restricted case that one gets a block of the proof that one should close off (because the block uses a special assumption $a \in A$), and the restricted case is much more common in actual proofs.
These rules are called “universal generalization”. The line reference would be to the block of statements from “Let \( a \in A \) be chosen arbitrarily” to \( P(a) \).

### 8.2 Using Universally Quantified Statements

If one is entitled to assume \((\forall x.P(x))\) and \(c\) is any name for an object, one is entitled to assume \(P(c)\).

If one is entitled to assume \((\forall x \in A.P(x))\) and \(c \in A\), one is entitled to assume \(P(c)\).

These rules are called “universal instantiation”. The reference is to the one or two previous lines used.

As rules of inference:

\[
\frac{(\forall x.P(x))}{P(c)}
\]

\[
\frac{(\forall x \in A.P(x))}{c \in A}
\]

\[
\frac{(\forall x \in A.P(x))}{c \in A}
\]

### 9 Existential Quantifier

This section presents rules for \((\exists x.P(x))\) (“for some \( x \), \( P(x) \)”, or equivalently “there exists an \( x \) such that \( P(x) \)” and for the restricted form \((\exists x \in A.P(x))\) (“for some \( x \) in the set \( A \), \( P(x) \)” or “there exists \( x \) in \( A \) such that \( P(x) \)”).

Notice that \((\exists x \in A.P(x))\) has just the rules one would expect from its logical equivalence to \((\exists x.x \in A \land P(x))\).

### 9.1 Proving Existentially Quantified Statements

To prove \((\exists x.P(x))\), find a name \( c \) such that \( P(c) \) can be proved. It is your responsibility to figure out which \( c \) will work.

To prove \((\exists x \in A.P(x))\) find a name \( c \) such that \( c \in A \) and \( P(c) \) can be proved. It is your responsibility to figure out what \( c \) will work.

A way of phrasing either kind of proof is to express the goal as “Find \( c \) such that \([c \in A \text{ and } P(c)]\)”, where \( c \) is a new name which does not appear in the context: once a specific term \( t \) is identified as the correct value of \( c \),
one can then say “let \( c = t \)” to signal that one has found the right object. Of course this usage only makes sense if \( c \) has no prior meaning.

This rule is called “existential introduction”. The reference is to the one or two lines used.

As rules of inference:

\[
\frac{P(c)}{(\exists x.P(x))} \quad \quad c \in A \quad \quad \frac{P(c)}{(\exists x \in A.P(x))}
\]

9.2 Using Existentially Quantified Statements

Suppose that one is entitled to assume \((\exists x.P(x))\) and one is trying to prove a goal \( C \). One is allowed to further assume \( P(w) \) where \( w \) is a name which does not appear in any earlier line of the proof that one is allowed to use, and prove the goal \( C \). Once the goal \( C \) is proved, one should no longer allow use of the block of variables in which the name \( w \) is declared (the reason for this is stylistic: one should be free to use the same variable \( w \) as a “witness” in a later part of the proof; this makes it safe to do so). If the statement one starts with is \((\exists x \in A.P(x))\) one may follow \( P(w) \) with the additional assumption \( w \in A \).

This rule is called “witness introduction”. The reference is to the line \((\exists x \in A.P(x))\) and the block of statements from \( P(w) \) to \( C \).

10 Proof Format

Given all these rules, what is a proof?

A proof is an argument which can be presented as a sequence of numbered statements. Each numbered statement is either justified by a list of earlier numbered statements and a rule of inference [for example, an appearance of \( B \) as line 17 might be justified by an appearance of \( A \) as line 3 and an appearance of \( A \rightarrow B \) as line 12, using the rule of modus ponens] or is an assumption with an associated goal (the goal is not a numbered statement but a comment). Each assumption is followed in the sequence by an appearance of the associated goal as a numbered statement, which we will call the resolution
of the assumption. The section of the proof consisting of an assumption, its resolution, and all the lines between them is closed off in the sense that no individual line in that section can be used to justify anything appearing in the proof after the resolution, nor can any assumption in that section be resolved by a line appearing in the proof after the resolution. In my preferred style of presenting these proofs, I will indent the section between an assumption and its resolution (and further indent smaller subsections within that section with their own assumptions and resolutions). The whole sequence of lines from the assumption to its resolution can be used to justify a later line (along with an appropriate rule of course): for example, the section of a proof between line 34: assume $A$: goal $B$ and line 71: $B$ could be used to justify line 113 $A \rightarrow B$ (lines 34-71, deduction ); I do not usually do this (I usually write the statement to be proved by a subsection as a goal at the head of that section, and I do not usually use statements proved in such subsections later in the proof), but it is permitted.

I usually omit the resolution of a goal if it is immediately preceded by an assumption-resolution section (or sections in the case of a biconditional) which can be used as its line justification: this seems like a pointless repetition of the goal, which will already appear just above such a section. I would state the resolution line if it was going to be referred to in a later line justification. The idea is that the statement of a goal followed by a block of text that proves it is accepted as a proof of that statement; the only reason to repeat the statement with a line number is if it is going to be referenced using that line number.

Note the important italicized phrase “can be”. A proof is generally presented in a mathematics book as a section of English text including math notation where needed. Some assumptions may be assumed to be understood by the reader. Some steps in reasoning may be omitted as “obvious”. The logical structure will not be indicated explicitly by devices like line numbering and indentation; the author will rely more on the reader understanding what he or she is writing. This means that it is actually quite hard to specify exactly what will be accepted as a proof; the best teacher here is experience. A fully formalized proof can be specified (even to the level where a computer can recognize one and sometimes generate one on its own), but such proofs are generally rather long-winded.