4. 13.4 Do one of the two parts. If you do both parts, your best work will count. If you do well on both parts, you might get some extra credit.

(a) Determine the curvature of the graph of \( y = x^2 \) at the point \((1, 1)\), the unit normal vector, and the radius of curvature. For a wee bit of extra credit, give the parameterization of the osculating circle.

Hint: The graph is parameterized by \((t, t^2, 0)\). The unit normal is much easier to find than usual because you know the plane that it lies in.

\[
\text{Curvature is } \frac{|f''(x)|}{\left(1 + (f'(x))^2\right)^{3/2}} = \frac{2}{\left(1 + (4x^2)^2\right)^{3/2}} = \frac{2}{17^{3/2}}
\]

so the radius of curvature is \( \frac{17^{3/2}}{2} \) (reciprocal of the curvature).

The tangent vector is \( \langle 1, 2t, 1 \rangle_{t=1} = \langle 1, 4, 0 \rangle \) so the unit normal is correctly \( \frac{\langle -4, 4, 1 \rangle }{\sqrt{1^2 + 4^2 + 1^2}} \) or \( \langle 4, -1, 0 \rangle \) the way the \( y = x^2 \) bends tells us it is pointing up so

\[
\frac{\langle -4, 4, 1 \rangle }{\sqrt{1^2 + 4^2 + 1^2}} = \frac{\langle -1, 1, 0 \rangle }{\sqrt{2}} \times \frac{1}{\sqrt{2}} \times \frac{1}{\sqrt{17}} \]

The center of the osculating circle is \( \langle 1, 1, 0 \rangle + \frac{17^{3/2}}{2} \langle -1, 1, 0 \rangle \times \frac{1}{\sqrt{2}} \times \frac{1}{\sqrt{17}} \)

\[
= \langle 1, 1, 0 \rangle + \frac{17^{3/2}}{2} \langle -4, 1, 0 \rangle = \langle 1, 0, 0 \rangle + \frac{17^{3/2}}{2} \langle -4, 1, 0 \rangle
\]

so the parameterization is

\[
\langle -33 + \frac{17^{3/2}}{2} \cos(\theta), \frac{19}{2}, \frac{17^{3/2}}{2} \sin(\theta), 0 \rangle
\]
(b) Determine the curvature of the curve parameterized by \( r(t) = \langle t, t^2, t^3 \rangle \) at the point \((2, 4, 8)\).

\[
\begin{align*}
\vec{r}'(t) &= \langle 1, 2t, 3t^2 \rangle \\
&= \langle 1, 4, 12 \rangle \\
\vec{r}''(t) &= \langle 0, 2, 6t \rangle \\
&= \langle 0, 2, 12 \rangle
\end{align*}
\]

\[
\left\| \langle 1, 4, 12 \rangle \times \langle 0, 2, 12 \rangle \right\| \\
= \left\| \langle 1, 4, 12 \rangle \right\|^3
\]

\[
\begin{align*}
\sqrt{24^2 + 2^2} \\
&= \sqrt{576 + 4} \\
&= \sqrt{580}
\end{align*}
\]

\[
\frac{\sqrt{580}}{\left(\sqrt{1^2 + 4^2 + 12^2}\right)^{\frac{3}{2}}}
\]

not a nice number.