Notes for Sections 12.1 through 12.7 (?)

Dr. Holmes

June 18, 2013

Contents

1 Section 12.1 1
2 Section 12.2 5
3 Section 12.3 7
4 Section 12.4 10
5 Section 12.5: Planes 13
6 Section 12.6: Quadric Surfaces 14
7 Section 12.7: Cylindrical and Spherical Coordinates 15

1 Section 12.1

1. First of all, I have a terminology issue with the book.

   The book says that a vector is an arrow from a point $P$ to a point $Q$. This is not really consistent with common mathematical usage, and in fact the book largely ignores this idea and is really talking about the same concept of vector as the rest of us.

   I will refer to arrows as arrows (where I can: it is unavoidable that we will refer to arrows as vectors in homework problems in the book). Arrow of the same length pointing in the same direction are pictures of the same vector, or we may call them copies of that vector.
2. In two dimensions, the vector from the point \((a, b)\) to the point \((c, d)\) is written \(\langle c - a, d - b \rangle\). The vector from \((1, 1)\) to \((3, 5)\) is \(\langle 2, 4 \rangle\), and it is the same vector as the vector from \((2, 3)\) to \((4, 7)\). Of course the arrow from \((1, 1)\) to \((3, 5)\) is not the same as the arrow from \((2, 3)\) to \((4, 7)\): each of these arrows is a translate of the other.

If \(P\) and \(Q\) are points, we may use the notation \(\overrightarrow{PQ}\) to denote the vector from \(P\) to \(Q\) (not the arrow). A variable representing a vector will be written \(\mathbf{v}\) in boldface or \(\mathbf{v}\) with an arrow over it. Arrows are easier on the board and boldface in printed text.

I do not say as the book does that vectors with the same length and direction are “equivalent”: they are simply the same vector. Arrows with the same length and direction are “equivalent” in the sense that they are pictures of the same vector.

3. The arrow from \(P\) to \(Q\) has \(P\) as its base and \(Q\) as its head. Each vector has a unique picture (arrow) with its head at the origin: considering this arrow can be referred to as “placing the vector in standard position”. The standard picture of any vector \(\langle a, b \rangle\) has its base at the origin and its head at \((a, b)\), setting up a natural correspondence between vectors in the plane and points in the plane.

4. Suppose \(P = (a, b)\) and \(Q = (c, d)\). The components of the vector \(\overrightarrow{PQ}\) are \(c - a\) (distance covered along the \(x\) direction) and \(d - b\) (distance covered along the \(y\) direction). The component notation \(\langle c - a, d - b \rangle\) is a name for this vector.

5. The length of a vector \(\langle u, v \rangle\) is \(\sqrt{u^2 + v^2}\) by the Pythagorean theorem: the length of the vector considered above is \(\sqrt{(c - a)^2 + (d - b)^2}\). We write the length of \(\mathbf{u}\) as \(|\mathbf{u}|\) (I do not at the moment know how to typeset the double bars the book uses, and this notation is also used). There is a zero vector \(\mathbf{0} = \langle 0, 0 \rangle\) which we assign length 0 and no direction (or any direction, depending on how you look at it).

6. We next consider vector algebra. I am not equipped here to draw pictures of vector addition, subtraction, and scalar multiplication; I will do this on the board.
Adding a vector \( u \) to a vector \( v \) can be represented using arrow by choosing an arrow representing \( u \), then drawing the unique arrow representing \( v \) whose base is at the head of the arrow chosen to represent \( u \): the arrow from the base of the given copy of \( u \) to the head of the given copy of \( v \) will be a picture of the vector \( u + v \), the sum of the two given vectors. The arrow from the head of the copy of \( v \) to the head of the copy of \( u \) (note the order!) will be a picture of \( u - v \), the difference of the vectors. Notice that the picture of \( u, v \), and \( u - v \) shows that \( v + (u - v) = u \), which is what we expect. [Pictures to be drawn on the board].

If \( u \) is a vector of length \( m \) and \( \lambda \) is a positive real number (which we call a scalar in this context), a picture of the vector \( \lambda u \) is an arrow pointing in the same direction as a picture of \( u \) and with length \( \lambda m \). \(-\lambda u\) is a vector of length \( \lambda m \) pointing in the opposite direction to that of \( u \). \( 0u \) is the zero vector. This operation on vectors is called scalar multiplication.

7. The operations just discussed are easy to compute from the component notation for vectors.
\[
\langle a, b \rangle + \langle c, d \rangle = \langle a + c, b + d \rangle \\
\langle a, b \rangle - \langle c, d \rangle = \langle a - c, b - d \rangle \\
\lambda \langle a, b \rangle = \langle \lambda a, \lambda b \rangle
\]

8. We state some familiar-looking algebraic properties of the vector operations (which are not the same as those for the operations on real numbers with the same names, because these are not really the same operations).
commutative: \( u + v = v + u \)

associative: \( u + (v + w) = (u + v) + w \)

distributivity of scalar multiplication over vector addition: \( \lambda (u + v) = \lambda u + \lambda v \)

identity: \( u + 0 = 0 + u = u \)

additive inverse: \( -u = (-1)u; u - v = u + (-v); u + (-u) = 0 \)

9. All of these properties can be verified for plane vectors by calculations using the properties of the real numbers. I will give examples.

A linear combination of vectors \( u \) and \( v \) is defined as any vector \( ru + sv \) where \( r \) and \( s \) are scalars.

If we are given a vector \( w \) and want to express it as a linear combination of vectors \( u \) and \( v \), this reduces to solving a system of linear equations (see example 4 in 12.1).

10. We have heard rumors that vectors are quantities with magnitude and direction, and we have seen how to compute length (magnitude) for a plane vector. We now show how to compute two different representations of the direction.

Where \( \lambda \) is a real number, we may write \( \frac{u}{\lambda} \) for \( \left( \frac{1}{\lambda} \right) u \).

For any vector \( u \), define \( e_u \) as \( \frac{u}{|u|} \). This will be the vector of length 1 pointing in the same direction as \( u \), and it is called the unit vector associated with \( u \). Vectors \( u \) and \( v \) have the same direction iff \( e_u = e_v \).

A vector whose standard picture has angle \( \theta \) from the positive \( x \) axis will be of the form \( \langle r \cos(\theta), r \sin(\theta) \rangle \), where \( r \) is the length of the vector.

This is familiar trigonometry content. We can then see that the unit vector associated with this vector has the form \( \langle \cos(\theta), \sin(\theta) \rangle \), so the angle \( \theta \) is readily computed from the unit vector (though one has to be careful: I may review this now, and certainly will when we need to do this kind of calculation).

11. Nonzero vectors are said to be parallel iff they have the same direction or exactly opposite directions, that is if one is a scalar multiple of the other (review why this is the same thing).

The zero vector is usually regarded as parallel to any vector.
12. The special notations $\mathbf{i}$ and $\mathbf{j}$ are used for the vectors $\langle 1,0 \rangle$ and $\langle 0,1 \rangle$.

13. **June 11 lecture will start here (I’ll present this example again).**
   I will do example 6, page 663.

14. We note the Triangle Inequality
   \[ |\mathbf{u} + \mathbf{v}| \leq |\mathbf{u}| + |\mathbf{v}|. \]
   Equality holds only if one of the vectors is the zero vector or if the vectors are parallel. (possibly review the relation of this to the actual triangle inequality in geometry, now or later).

## 2 Section 12.2

1. In this section we consider vectors in three dimensional space instead of in the plane. We assume the use of three coordinate axes which are pairwise perpendicular and satisfy the right hand rule.

2. The distance formula in 3-space should be familiar.

3. I’ll review the equations of spheres and cylinders in the book.

4. The relations between points, arrow and vectors in 3-space are the same as in 2-space. An arrow from $(a,b,c)$ to $(d,e,f)$ is a picture of the vector $(d-a,e-b,f-c)$. The vector has length $\sqrt{(d-a)^2 + (e-b)^2 + (f-c)^2}$.

5. Computations of sums differences and scalar multiples of vectors given in component notation are computed in the same way in 3-space as in the plane. Unit vectors are computed in the same way (“divide” a vector by its length).

6. The really new idea in this section is the description of parametric equations of lines in 3-space.

   The line through $(x_0,y_0,z_0)$ in the direction of the vector $(a,b,c)$ can be given the vector parameterization $\mathbf{r}(t) = \langle x_0, y_0, z_0 \rangle + t \langle a, b, c \rangle$
   or the coordinatewise parametric equations $x = x_0 + at; y = y_0 + bt; z = z_0 + ct$.

   Why does this work? If $(x,y,z)$ is a point on the line, then the vector $\langle x-x_0, y-y_0, z-z_0 \rangle$ from $(x_0,y_0,z_0)$ to $(x,y,z)$ is parallel to $(a,b,c)$,
so is a scalar multiple, thus for some real $t$ we have

$$\langle x - x_0, y - y_0, z - z_0 \rangle = t \langle a, b, c \rangle,$$

whence

$$\langle x, y, z \rangle - \langle x_0, y_0, z_0 \rangle = t \langle a, b, c \rangle$$

and $\langle x, y, z \rangle = \langle x_0, y_0, z_0 \rangle + t \langle a, b, c \rangle$ as desired. This shows that every point on this line is of the form

$$\langle x_0, y_0, z_0 \rangle + t \langle a, b, c \rangle;$$

we can argue very similarly that every point of this form is on the line (here we are confusing points with vectors in standard position).

7. The meaning of these equations is that the points on the line are exactly the points obtained by plugging in values of $t$ ranging over all the reals into these equations. (Notice that for the first one we have to pass from a vector to the associated point, the head of the picture of that vector in standard position)

8. Different lines can have the same parameterization. Do Example 5, 12.2.

9. We can use the parameterization and algebra to find intersections. Do Example 6. Do problem 50 or something similar.

10. We can also generate a parameterization from two points on a line, which is often useful.

The line through $(a, b, c)$ and $(d, e, f)$ has the vector parameterization

$$\mathbf{r}(t) = (1 - t) \langle a, b, c \rangle + t \langle d, e, f \rangle$$

and coordinatewise parameterizations $x = a + (d - a)t; y = b + (e - b)t; z = c + (f - c)t$

I will explain why this is actually an example of the previously given parameterization of a line: the line has the point $(a, b, c)$ on it and is parallel to the vector $\langle d - a, e - b, f - c \rangle$, so standard vectors with heads on this line take the form $\langle a, b, c \rangle + t \langle d - a, e - b, f - c \rangle = (1 - t) \langle a, b, c \rangle + t \langle d, e, f \rangle$
3 Section 12.3

1. The dot product \( \langle a, b, c \rangle \cdot \langle d, e, f \rangle \) is defined as \( ad + be + cf \). Notice that the dot product of two vectors is a scalar; this is often called the scalar product.

The definition for vectors in 2-space is similar. \( \langle a, b \rangle \cdot \langle d, e \rangle \) is defined as \( ad + be \).

2. There are familiar-looking algebraic properties of the dot product (if one thinks of it as a kind of multiplication). But the resemblances are limited.

   - zero property: \( 0 \cdot v = v \cdot 0 = 0 \)
   - commutativity: \( u \cdot v = v \cdot u \)
   - pulling out scalars (associativity with scalar multiplication, sort of): \( (\lambda v) \cdot w = v \cdot (\lambda w) = \lambda (v \cdot w) \)
   - distributivity: \( u \cdot (v + w) = u \cdot v + u \cdot w \) (this form of the law with commutativity of the dot product is enough to show the other form given in the book).

   relation to length: \( v \cdot v = |v|^2 \)

   I’ll give verifications of some of these. It is a good idea to try verifying all of them yourself.

3. Dot products have a very important relation to angles between vectors.

\[
v \cdot w = |v||w| \cos(\theta)
\]

where \( \theta \) is the angle between the vectors.

This is proved using the Law of Cosines.

\[
|v - w|^2 = |v|^2 + |w|^2 - 2|v||w| \cos(\theta)
\]

is true by the Law of Cosines (draw a picture of the vectors involved).

By algebraic properties,

\[
|v - w|^2 = |v|^2 + |w|^2 - 2(v \cdot w)
\]
– to show this, expand
\[ |v - w|^2 = (v - w) \cdot (v - w) \]
using the algebraic properties, and we get
\[ v \cdot v + w \cdot w - 2(v \cdot w), \]
that is
\[ |v|^2 + |w|^2 - 2(v \cdot w). \]
Since we have
\[ |v - w|^2 = |v|^2 + |w|^2 - 2|v||w| \cos(\theta) = |v|^2 + |w|^2 - 2(v \cdot w), \]
we see by algebra that
\[ v \cdot w = |v||w| \cos(\theta) \]

4. We can use this formula to compute the angle between two vectors. We always assume that the angle between two vectors is between 0 and \( \pi \) radians inclusive. The formula above and algebra give
\[ \cos(\theta) = \frac{v \cdot w}{|v||w|} \]
The face that the arc cos of a number is the unique angle between 0 and \( \pi \) with that number as its cosine, and our choice of the angle in this range as \( \theta \), allow us to further state
\[ \theta = \arccos \left( \frac{v \cdot w}{|v||w|} \right) \]
We can compute the exact cosine of the angle between two vectors given in component form (usually involving radicals in the lengths in the denominator) then use the arccos function on our calculator to get a good value for the angle between the two vectors.
5. There is a very important special case of angle computations, constantly useful in theoretical and practical reasoning. The angle between $v$ and $w$ is a right angle iff $v \cdot w = 0$. This fact will be used constantly. Vectors with this property are said to be perpendicular or orthogonal. We also write $v \perp w$ in this situation.

6. We now discuss projections and components. I do not like the author’s notation because it leaves out the vector $v$.

The projection of $u$ onto $v$ is the vector $(u \cdot e_v)e_v$ (notice that this is a scalar product of a dot product and a unit vector) or

$$\left(\frac{u \cdot v}{v \cdot v}\right)v$$

(check that these are the same). I will write this $\text{proj}_v(u)$.

The scalar $u \cdot e_v$, which is the length of $\text{proj}_v(u)$, is called the component of $u$ in the direction of $v$, and I will write it $\text{comp}_v(u)$.

It is an important fact that

$$\text{proj}_v(u) \perp u - \text{proj}_v(u)$$

: these two vectors give a decomposition of $u$ into the sum of two vectors, one parallel to $v$ and one perpendicular to $v$. $\text{proj}_v(u)$ is parallel to $v$ because it is a scalar multiple of $v$.

$$v \cdot (u - \text{proj}_v(u)) = v \cdot u - v \cdot \left(\frac{u \cdot v}{v \cdot v}\right)v =$$

$$v \cdot u - \left(\frac{u \cdot v}{v \cdot v}\right)(v \cdot v) = v \cdot u - u \cdot v = 0$$

This calculation shows that $v$ is perpendicular to $u - \text{proj}_v(u)$: since $\text{proj}_v(u)$ is parallel to $v$ it is also perpendicular to $u - \text{proj}_v(u)$.

I plan to do examples 6 and 7 and possibly example 8.
4 Section 12.4

1. This section is all about the definition and properties of a rather mysterious operation on vectors in three dimensional space called the **cross product**. Once we have defined this, we will have *three* different forms of “multiplication” involving vectors, namely scalar multiplication (a real “times” a vector yields a vector), the dot product (a vector “times” a vector yields a real) and the cross product (a vector “times” a vector yields a vector).

2. In order to make the computational definition of the cross product easier to digest, we first define two by two and three by three determinants. The definition of three by three determinants that I give is not the same as the one given in the book (though you can check with some effort that it is the same operation that is defined).

   The definition given in the book has the advantage that it generalizes to determinants of higher order (4 or more). The definition for 3 by 3 determinants given here does not generalize to higher orders, but we have no need to compute determinants of higher orders than 3 in this class.

   Two by two determinants:

   \[
   \begin{vmatrix}
   a & b \\
   c & d \\
   \end{vmatrix}
   \]

   is defined as

   \[ad - bc.\]

   Three by three determinants:

   \[
   \begin{vmatrix}
   a & b & c \\
   d & e & f \\
   g & h & i \\
   \end{vmatrix}
   \]

   is computed in a way best explained by copying over the first two columns to get this array:

   \[
   \begin{vmatrix}
   a & b & c | a & b \\
   d & e & f | d & e \\
   g & h & i | g & h \\
   \end{vmatrix}
   \]
then adding up the products of all the downward diagonals and the additive inverses of the products of all the upward diagonals:

\[ aei + bfg + cdh - gec - hfa - idb \]

3. Recall the symbols \( \mathbf{i} \mathbf{j} \mathbf{k} \) for the vectors \( \langle 1, 0, 0 \rangle, \langle 0, 1, 0 \rangle, \langle 0, 0, 1 \rangle \). We define the cross product purely formally as a three by three determinant – formally because some of the entries are vectors and the definition of three by three determinant really only applies to three by three arrays of real numbers. \( \langle a, b, c \rangle \times \langle d, e, f \rangle \) is defined as

\[
\begin{vmatrix}
  \mathbf{i} & \mathbf{j} & \mathbf{k} \\
  a & b & c \\
  d & e & f
\end{vmatrix}
\]

which expands out to

\[(bf - ce)i + (cd - af)j + (ae - bd)k\]

which can also be written

\[
\begin{vmatrix}
  b & c \\
  e & f
\end{vmatrix}
\begin{vmatrix}
  \mathbf{i} & a & c \\
  \mathbf{j} & d & f
\end{vmatrix}
\begin{vmatrix}
  a & b \\
  d & e
\end{vmatrix}
\mathbf{k}
\]

4. Important algebraic properties of the cross product are given. They can be verified by computations from the definition. One of them is quite unfamiliar: for any vectors \( \mathbf{u} \) and \( \mathbf{v} \), \( \mathbf{u} \times \mathbf{v} = -(\mathbf{v} \times \mathbf{u}) \): this operation is anticommutative.

**anticommutativity:** \( \mathbf{u} \times \mathbf{v} = -(\mathbf{v} \times \mathbf{u}) \)

**nilpotence:** \( \mathbf{u} \times \mathbf{u} = \mathbf{0} \). The name given is unimportant (I just needed a name for it): this is a consequence of anticommutativity.

**converse of nilpotence:** \( \mathbf{u} \times \mathbf{v} = \mathbf{0} \) iff \( \mathbf{u} \) is \( \mathbf{0} \) or \( \mathbf{v} \) is a scalar multiple of \( \mathbf{u} \). Again, the name I give for the property doesn’t matter.

**relation to scalar multiplication:** \( \lambda(\mathbf{u} \times \mathbf{v}) = (\lambda\mathbf{u}) \times \mathbf{v} = \mathbf{u} \times (\lambda\mathbf{v}) \)
\section*{distributivity: $u \times (v + w) = u \times v + u \times w$}

\[(u + v) \times w = u \times w + v \times w\]

Strictly speaking, each of the versions of the distributive law is enough by itself, as in combination with anticommutativity and the relation of the cross product to scalar multiplication, each version can be deduced from the other.

5. The cross product of two vectors can be computed using the algebraic properties and the products of the basis vectors $i \ j \ k$. We can verify that $i \times j = k$, $j \times k = i$, $k \times i = j$: the other products of basis vectors can be deduced from anticommutativity and nilpotence. Then the cross product of any two vectors can be determined by expressing the vectors as linear combinations of the basis vectors in the obvious way ($\langle a, b, c \rangle = ai + bj + ck$) and applying algebra in an almost familiar way (the unfamiliar bit is the use of anticommutativity of the cross product).

I’ll do a computational example.

6. The cross product $u \times v$ is perpendicular to each of $u$ and $v$.

To see this, show that $u \cdot (u \times v) = 0$.

If $u = \langle a, b, c \rangle$ and $v = \langle d, e, f \rangle$ then

$$u \cdot (u \times v) = \langle a, b, c \rangle \cdot \langle bf - ce, cd - af, ae - bd \rangle$$

$$= abf - ace + bcd - baf + cae - cbd = 0.$$

Then $v \cdot (u \times v) = -v \cdot (v \times u) = 0$ by anticommutativity followed by the same computation as above with the two vectors interchanged.

7. The length of $u \times v$ is $|u||v|\sin(\theta)$.

The book verifies this by first proving the identity

$$|u \times v|^2 = |u|^2|v|^2 - (u \cdot v)^2,$$

which can be done by applying the definitions and horrible algebra.

Now

$$|u|^2|v|^2 - (u \cdot v)^2 = |u|^2|v|^2 - |u|^2|v|^2 \cos^2(\theta) = |u|^2|v|^2 \sin^2(\theta)$$

The sine of $\theta$ is known to be positive so $|u \times v| = |u||v|\sin(\theta)$ follows.
8. These two pieces of information almost give a complete geometrical description of \( \mathbf{u} \times \mathbf{v} \): the added information is which of the two vectors perpendicular to both \( \mathbf{u} \) and \( \mathbf{v} \) and of the right length one should choose, and the short answer (which we will not verify) is that \( \mathbf{u}, \mathbf{v}, \) and \( \mathbf{u} \times \mathbf{v} \) form a right-handed system (if the fingers of your right hand curl from \( \mathbf{u} \) to \( \mathbf{v} \) then your thumb will point to the side of the plane the two vectors lie in on which \( \mathbf{u} \times \mathbf{v} \) is found.

9. There are useful relationships between the cross product and areas of parallelograms and volumes of parallelepipeds, to be inserted.

The area of the parallelogram determined by \( \mathbf{u} \) and \( \mathbf{v} \) is the length of the cross product of the two vectors.

The volume of the parallelepiped determined by \( \mathbf{u} \mathbf{v} \) and \( \mathbf{w} \) is the absolute value of \( \mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) \) (this real quantity might be negative).

5 \hspace{1em} \text{Section 12.5: Planes}

Everything in this section hinges on a single idea, combined with the powerful computational capabilities afforded by the dot and cross products.

A plane is determined by a point on the plane and a vector \textit{perpendicular} to the plane.

If \((u, v, w)\) is a point on a plane \(P\) and \(\langle a, b, c \rangle\) is perpendicular to \(P\), then we know for any point \((x, y, z)\) on the plane that the vector from \((u, v, w)\) to \((x, y, z)\) is perpendicular to \(\langle a, b, c \rangle\).

The vector from \((u, v, w)\) to \((x, y, z)\) is \(\langle x - u, y - v, z - w \rangle\), and we get the equation \(\langle x - u, y - v, z - w \rangle \cdot \langle a, b, c \rangle = 0\) by the relationship between the dot product and perpendicularity. This is equivalent to the scalar equation \(a(x - u) + b(y - v) + c(z - w) = 0\), which can be converted to the form \(ax + by + cz = d\) by algebra.

The vector perpendicular to a plane is called a normal vector to the plane; this is why the book uses \(\mathbf{n}\) as the typical letter for a normal vector. Notice that we have a vector form for the equation of a plane and two different scalar forms:

\[
\mathbf{n} \cdot \langle x, y, z \rangle = d
\]

\[
ax + by + cz = d
\]
\[ a(x - u) + b(y - v) + c(z - w) = 0 \]

do Examples 1 and 2. Note that parallel planes have the same normal vector; also note that we can read the normal vector right off either of the scalar equations for the plane.

Example 3 introduces a further idea: suppose that we are given three points in a plane: from this we can easily get two vectors parallel to the plane (if the three points are not on the same line); or we might alternatively be given a point on a plane and two vectors parallel to it. The trick we use to get an equation for the plane is to get a vector perpendicular to the plane by taking the cross product of the two given vectors parallel to the plane.

The trace of a plane in a coordinate plane or a plane parallel to one of the coordinate planes is the intersection of the two planes, which is in this case easily determined by setting one of the variables to 0 or to a fixed value.

A final important idea to use in your homework is that the angle between two planes is the same as the angle between their normal vectors.

problems I might do in lecture: 1, 19, 27, 29, 30, 33, 41, 51, 57

6 Section 12.6: Quadric Surfaces

This is really something I want to touch on quickly, but not spend much time on.

Recall that \( ax^2 + by^2 = c \) is an ellipse centered at the origin, and \( ax^2 - by^2 = c \) and \( ay^2 - bx^2 = c \) are hyperbolas centered at the origin. The first intersects the \( y \)-axis but not the \( x \)-axis (draw a picture and note the algebraic reasons); the second intersects the \( x \)-axis and not the \( y \)-axis (draw a picture and note the algebraic reasons).

In this section we consider various surfaces with second order equations. We only consider surfaces which are nicely orientd with respect to the axes, and we can classify them by looking at their traces.

\( ax^2 + by^2 + cz^2 = d \) will be an ellipsoid.

\( ax^2 \pm by^2 \pm cz^2 = d \) with one or both signs negative gives a hyperboloid. If one sign is negative, it is a hyperboloid of one sheet (hyperbola traces in two directions, ellipse traces in one direction, all elliptical traces nonempty); if it has both signs negative then it is a hyperboloid of two sheets (hyperboloid traces on two coordinate planes, elliptical traces on one coordinate plane,
some elliptical traces are empty). Explain the algebra, explore the geometry, show that the forms they give are equivalent.

Take a look at the equations of paraboloids and cones and discuss their traces.

7 Section 12.7: Cylindrical and Spherical Coordinates

pending...will begin with a review of polar coordinates.

Polar coordinates \((r, \theta)\) with \(r \geq 0\) represent a point at distance \(r\) from the origin in a direction at angle \(\theta\) from the positive \(x\) axis working counterclockwise. Notice that adding \(2k\pi\) for \(k\) an integer will not change what point is represented.

If \(r < 0\), the point is at distance \(|r|\) from the origin in the direction opposite to the ray at angle \(\theta\) with the positive \(x\) axis. The point \((-r, \theta)\) is the same as the point \((r, \theta + \pi)\) (adding a straight angle reverses direction).

The cartesian coordinates for the point with polar coordinates \((r, \theta)\) are \((r \cos(\theta), \sin(\theta))\).

If we are given cartesian coordinates, the polar coordinates are not uniquely determined, but they will satisfy \(x^2 + y^2 = r^2\) and \(\tan(\theta) = \frac{y}{x}\). Resist the temptation to declare \(\theta\) equal to \(\arctan\left(\frac{y}{x}\right)\); this may be modified by what quadrant you are in (though you can always make it work if willing to let \(r\) be negative).

When polar coordinates are used for graphing, \(r\) is expressed as a function of \(\theta\). There is an obvious prejudice in favor of periodic functions in such graphing!

Cylindrical coordinates \((r, \theta, z)\) describe a point with cartesian coordinate \(z\) (geometrically \(z\) is the distance from the point to the \(xy\) plane. \(r\) and \(\theta\) are the polar coordinates of the projection of the point to the \(xy\) plane.

The point with cylindrical coordinates \((r, \theta, z)\) has cartesian coordinates \((r \cos(\theta), r \sin(\theta), z)\).

The point with cartesian coordinates \((x, y, z)\) has many sets of polar coordinates: they will satisfy the conditions \(r^2 = x^2 + y^2\), \(\tan(\theta) = \frac{y}{x}\), \(z = z\). Remember that \(r\) is not strictly analogous to the polar coordinate \(r\) as it is not the distance from the origin to \((x, y, z)\), but the distance from the origin to its projection \((x, y, 0)\) on the \(xy\) plane.
Presumably there might be situations where one would want to use cylindrical coordinates on a different pair of axes.

When one graphs functions of two variables in cylindrical coordinates, one expresses $r$ as a function of $\theta$ and $z$: an equation in $x, y, z$ can be converted into an equation in $r, \theta, z$ using the conditions above, then solved for $r$ in terms of the other variables.

Spherical coordinates $(\rho, \theta, \phi)$ describe a point at distance $\rho$ from the origin, with the same coordinate $\theta$ that it has in cylindrical coordinates, and with $\phi$ equal to the angle between 0 and $\pi$ radians that the vector from the origin to the point makes with the positive $x$ axis. Notice that $\theta$ takes values all the way from 0 to $2\pi$ meaningfully (or $-\pi$ to $\pi$ if you prefer) but $\phi$ is more restricted.

The cylindrical component $r$ of the point with spherical coordinates $(\rho, \theta, \phi)$ satisfies $r = \rho \sin(\phi)$ (immediate by trigonometry). Similarly the $z$ coordinate of the point is $\cos(\phi)$. Now $x = r \cos(\theta) = \rho \sin(\phi) \cos(\theta)$ and $y = r \sin(\theta) = \rho \sin(\phi) \sin(\theta)$.

The point with cartesian coordinates $(x, y, z)$ has many different possible assignments of spherical coordinates: they will all satisfy the conditions $\rho^2 = x^2 + y^2 + z^2$; $\tan(\theta) = \frac{y}{x}$; $\cos(\phi) = \frac{z}{\rho}$.

Test questions on this material will be of the same kinds shown in the homework, and formulas will be supplied.