Notes on Assignment VII, problem 2

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The theorem to be proved is "for all $n \geq 2$, $F_{n-1}F_{n+1} = F_n^2 + (-1)^n$".

I suggested rephrasing this to

"for all n, $F_{2n-1}F_{2n+1} = F_{2n}^2 + 1$ and $F_{2n}F_{2n+2} = F_{2n+1}^2 - 1$"; this separates out the odd and even cases.

The basis step is easy.

The induction hypothesis is $F_{2k-1}F_{2k+1} = F_{2k}^2 + 1$ and $F_{2k}F_{2k+2} = F_{2k+1}^2 - 1$.

It is important to notice that this implies that $F_{2k}^2 = F_{2k-1}F_{2k+1} - 1$ and $F_{2k+1}^2 = F_{2k}F_{2k+2} + 1$!

From this I am supposed to prove $F_{2k+1}F_{2k+3} = F_{2k+2}^2 + 1$ and $F_{2k+2}F_{2k+4} = F_{2k+3}^2 - 1$. (can you see why?)

This turns out to be a good deal easier than I was making it. It may not even be necessary to change the form of the statement to get a good proof.

For example, to prove $F_{2k+1}F_{2k+3} = F_{2k+2}^2 + 1$ proceed as follows:

$$F_{2k+1}F_{2k+3}$$

= (def of Fibonacci numbers)

$$F_{2k+1}(F_{2k+1} + F_{2k+2})$$

= (algebra)

$$F_{2k+1}^2 + F_{2k+1}F_{2k+2}$$

= (ind hyp)

$$(F_{2k}F_{2k+2} + 1) + F_{2k+1}F_{2k+2}$$

= (algebra)

$$(F_{2k} + F_{2k+1})F_{2k+2} + 1$$

= (def of Fibonacci numbers)

$$F_{2k+2}^2 + 1$$

which is what we want.

You could complete this writeup (adding the similar proof that $F_{2k+2}F_{2k+4} = F_{2k+3}^2 - 1$), but I think that it is actually possible to prove the original statement by ordinary induction directly by adapting my calculations in the right way (it seems that this is an "easy" theorem about the Fibonacci numbers that doesn't require a special variant of induction!)

i.e.,

prove $F_1F_3 = F_2^2 + (-1)^1$ (basis step, $k = 2$)

then prove "If $k \geq 2$ and $F_{k-1}F_{k+1} = F_k^2 + (-1)^k$, then $F_kF_{k+2} = F_{k+1}^2 + (-1)^{k-1}$.

The calculation above can be adapted to prove this induction step (if you think just a little about the powers of -1); just remember that the calculation above has $2k$ instead of k (and plain 1 instead of $(-1)^{2k}$).