Additional Notes for February 3

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These notes discuss additional content in the February 3 lecture which is not found in the book.

Additional rules for summation:

basic properties:

$$\Sigma_{i=1}^{1} f(i) = f(1)$$

$$\Sigma_{i=1}^{n+1} f(i) = (\Sigma_{i=1}^{n} f(i)) + f(n+1)$$

summing units:

$$\Sigma_{i=1}^{n} 1 = n$$

distributing a constant multiplier:

$$c\Sigma_{i=1}^{n} f(i) = \Sigma_{i=1}^{n} cf(i)$$

adding summations with the same range:

$$\Sigma_{i=1}^{n} f(i) + \Sigma_{i=1}^{n} g(i) = \Sigma_{i=1}^{n} (f(i) + g(i))$$

telescoping sums:

$$\Sigma_{i=1}^{n} (f(i+1) - f(i)) = f(n+1) - f(1)$$

Development of the formula

$$\Sigma_{i=1}^{n} i = \frac{n(n+1)}{2}$$

:

By telescoping sums,

$$\Sigma_{i=1}^{n}((i+1)^2 - i^2) = (n+1)^2 - 1^2 = n^2 + 2n + 1 - 1 = n^2 + 2n.$$

By algebra and by applying rules for summations given above,

$$\Sigma_{i=1}^{n}((i+1)^2-i^2) = \Sigma_{i=1}^{n}(i^2+2i+1-i^2) = \Sigma_{i=1}^{n}(2i+1) = 2(\Sigma_{i=1}^{n} i)\Sigma_{i=1}^{n} 1 = 2(\Sigma_{i=1}^{n} i)+n.$$

Things equal to the same thing are equal to each other, so

$$2(\Sigma_{i=1}^{n} i) + n = n^2 + 2n,$$

and so

$$2(\Sigma_{i=1}^{n} i) = n^2 + 2n - n = n^2 + n,$$

and so

$$(\Sigma_{i=1}^{n} i) = \frac{n^2 + n}{2} = \frac{n(n+1)}{2}.$$

Here we have proved the formula, but not by induction. Note that we have not only proved the formula, but we have developed it. This won't be true with the induction proof.

Prove the previous formula by induction:

Theorem:

$$(\Sigma_{i=1}^{n} i) = \frac{n(n+1)}{2}$$

Proof: The proof is subdivided into the basis step and induction step.

Basis step: The goal is to prove

$$(\Sigma_{i=1}^{1} i) = \frac{1(1+1)}{2}$$

.

$$(\Sigma_{i=1}^{1} i) = 1 = \frac{1(1+1)}{2}$$

is true.

Induction step: The goal is to prove that for any natural number k, if $(\Sigma_{i=1}^{k} i) = \frac{k(k+1)}{2}$ then $(\Sigma_{i=1}^{k+1} i) = \frac{(k+1)(k+2)}{2}$.

Let k be an arbitrary natural number. (the proof of a universal sentence about natural numbers will usually start this way).

Assume $(\Sigma_{i=1}^{k} i) = \frac{k(k+1)}{2}$. This is the *inductive hypothesis*. You should always clearly indicate the inductive hypothesis in your induction proofs and indicate clearly where it is used. Our goal is now to prove $(\Sigma_{i=1}^{k+1} i) = \frac{(k+1)(k+2)}{2}$.

$$(\Sigma_{i=1}^{k+1} i) = (\Sigma_{i=1}^{k} i) + (k+1)$$

by a basic property of summations

$$= \frac{k(k+1)}{2} + k + 1$$

by inductive hypothesis and substitution of equals for equals

$$= \frac{k^2+k}{2} + \frac{2k+2}{2} = \frac{k^2+3k+2}{2} = \frac{(k+1)(k+2)}{2}$$

by algebra, which completes the proof.

If you look at this file again later, you may find the derivation of the formula for $\Sigma_{i+1}^{n} i^2$ from the rules for summations inserted here.