Defining the Real Numbers

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This is an optional set of notes on defining the real numbers in the same spirit in which we have defined the integers and rationals.

We will define the *positive* reals assuming that we have already defined the *positive* rationals.

We defined the integers by considering equivalence classes of pairs (m, n) of natural numbers used to represent the results of the subtraction problem $m - n$ in the integers: the reason the construction works is that we can say what it means for the integer $m - n$ to be equal to the integer $p - q$, where m, n, p, q are natural numbers, entirely in terms of natural numbers.

Similarly, we define the rationals by considering equivalence classes of pairs (m, n) of an integer and a positive integer used to represent fractions $\frac{m}{n}$. We already know from elementary school how to express what it means for $\frac{m}{n}$ to be equal to $\frac{p}{q}$ entirely in terms of integers $(mn = pq)$ and the rest goes from there.

The new operation we add to get the reals is *infinite sums* over the rationals. We stick to infinite sums of *positive* rationals in our basic definition because there are technical problems with infinite sums involving both positive and negative rationals (they can be undefined in more complicated ways).

We use infinite sets S of positive rational numbers to represent infinite sums. A sum of finitely many elements of S is called a "partial sum" of S.

We say that the sum of S exists (is finite) if there is a positive rational b such that every partial sum of S is less than b.

If the sum of S exists and the sum of T exists, we say that the sum of S is less than or equal to the sum of T if for any partial sum of S there is a partial sum of T which is greater. We say that the sums of S and T are the same if the sum of S is less than or equal to the sum of T (using the definition above) and the sum of T is less than or equal to the sum of S.

The relation between S and T defined by "the sum of S is the same as the sum of T" as we have just defined it is an equivalence relation. (this is not too hard to prove). The equivalence classes under this relation of the sets whose sums exist can be used to represent the positive real numbers.

Notice that an individual real number codes an infinite amount of information (this is not true for individual natural numbers, integers, or rationals).

Notice that any infinite decimal representing a positive real can be converted to an infinite set of positive rationals in a straightforward way – the infinite decimal representation maps to our presentation. The terminating decimals cannot be converted to infinite sums of positive rationals, but every terminating decimal is equivalent to an infinite decimal: lower the last nonzero digit by 1 and replace all following digits by 9's ($1.5 = 1.49999\ldots$).

There is a technical problem with this definition: it only handles infinite sums of positive rationals which are all different! This makes it harder to define addition and multiplication in a natural way, though it does not make it impossible. It would be natural to define $[S] + [T]$ as the equivalence class of the union of S and T, but this is made difficult by the fact that if S and T have members in common this will cause the sum to come out too small (because a repeated term in the sum will be lost). There are ways to get around this, either by defining addition cleverly using our definition of the positive reals or by changing the definition of the positive reals in such a way as to allow repeated terms in infinite sums. For example, if we used sums of infinite sequences of positive rationals instead of sums of infinite sets there would be no problem with repeated terms.

To get zero and the negative reals, construct the general real numbers from the positive reals in the same way we constructed the integers from the natural numbers.

So the order of construction will be different: from the natural numbers, build the positive rationals without first building the integers (ordered pairs of natural numbers under the equivalence relation of having the same rational quotient). Then build the positive real numbers as infinite sums of rationals. Then build the reals as ordered pairs of positive reals under the relation of having the same difference.