Math 187 Assignment VII

Dr. Holmes

February 12, 2004

1. Prove by induction that every natural number ≥ 9 can be expressed in the form $2a + 5b$, where a and b are natural numbers.

   Hint: I suggest proving it by strong induction as follows. Divide this into two cases. Show that it is true for each number from 9 to 17 separately (treat all of these as basis steps). Then the strong induction step will handle all numbers ≥ 18.

   If you can prove it by induction in some other way, this is also acceptable (I can see at least one other approach).

2. Prove by induction (the form is up to you, but you should have some idea what is likely to work with Fibonacci numbers) that $(f_{n-1})(f_{n+1}) = f_n^2 + (-1)^n$, for each $n \geq 2$ (where f_n is the nth Fibonacci number, as usual).

   Remember that you can express any f_k (with $k > 2$) in the form $f_{k-1} + f_{k-2}$.

   I think this problem is quite hard; I will make some hints available, and I encourage you to come to my office if you have questions about this one.

3. Let $f : \mathcal{N} \rightarrow \mathcal{N}$ be defined recursively by

$$f(1) = 7;$$

$$f(n+1) = \frac{n}{2}$$

   if $f(n)$ is even;

$$f(n+1) = 3f(n) + 1$$

if $f(n)$ is odd. Explain in detail why f is not an injection and not a surjection.

This should not be hard: it is very similar to an example we did in class.

4. Let $f : \mathcal{N} \to \mathcal{N}$ and $g : \mathcal{N} \to \mathcal{N}$ be defined by $f(n) = 2n + 1$ and $g(n) = n^2 + 1$. Present simplified formulas for the compositions fg and gf.

   It is hard to define functions from the naturals to the naturals which have inverses. Extend each of the functions f and g given above to the function from the real numbers to the real numbers with the same formula. For each function, if it has an inverse, determine a formula for the inverse function, and if it does not, explain why it does not.

   Give a description of a function from the natural numbers to the natural numbers which is not the identity function but which is its own inverse.

   Much of this problem should be review of what you already know about compositions and inverses.

5. Construct an example of functions $f : A \to B$ and $g : B \to A$ such that f is an injection, g is not an injection, but gf is still an injection. You can use finite sets A and B (arrow diagrams are acceptable). Is this possible if f is a bijection? If it is possible, give an example. If it is not possible, prove that it is not possible.

   I will do a similar problem in class as a model for thinking about this problem.