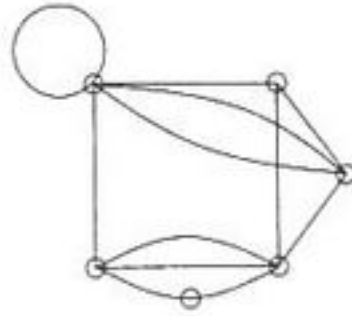
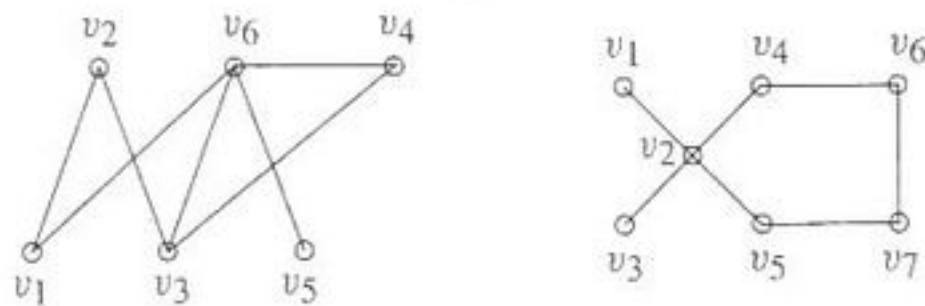


4. (a) How many vertices and how many edges does the pseudograph contain? What is the degree sequence of this pseudograph?



- (b) Verify Proposition 9.2.5 and Corollary 9.2.7 for this pseudograph.
5. [BB] Draw a graph with five vertices and as many edges as possible. How many edges does your graph contain? What is the name of this graph and how is it denoted?
6. (a) What is the maximum degree of a vertex in a graph with n vertices?
 (b) What is the maximum number of edges in a graph with n vertices?
 (c) Given a natural number n , does there exist a graph with n vertices and the maximum possible number of edges?
7. Draw \mathcal{K}_7 , $\mathcal{K}_{3,4}$, and $\mathcal{K}_{2,6}$.
8. Draw a graph with 64 vertices representing the squares of a chessboard. Connect two vertices with an edge if you can move legally between the corresponding squares with a single move of a knight. [The moves of a knight are L-shaped, two squares vertically (or horizontally) followed by one square horizontally (respectively, vertically).]
- (a) Explain why this graph is bipartite.
 (b) What are the degrees of the vertices?
9. Consider again the graphs accompanying Exercise 5 of Section 9.1, which we reproduce here.

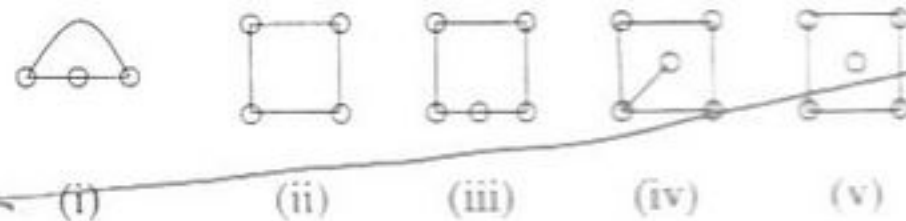


- (a) [BB] For the graph on the left,
- i. Make a table which shows the least number of edges joining each pair of vertices in this graph. (Such a table displays the least number of stops required on air trips between cities in the region depicted by the graph.)

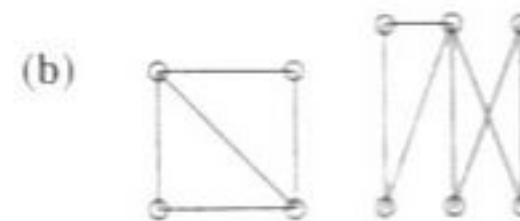
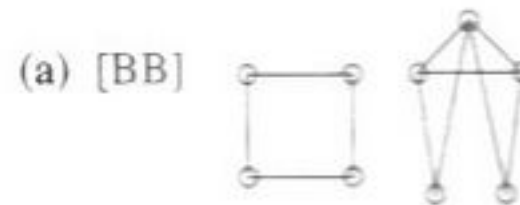
- ii. Add up the numbers in each column of the table. Divide each column total by the degree of the corresponding vertex. These ratios are called *accessibility indices* since they measure the relative accessibility of the cities (by air). Which city is the most accessible? Which is the least accessible?
- iii. Suppose a direct flight joining cities v_1 and v_3 is introduced. What is the new beta index of the graph? What are the new accessibility indices? Which city is most accessible now? Which city is now least accessible?
- iv. Repeat part (iii), assuming a flight is introduced between cities v_2 and v_6 instead of between v_1 and v_3 .

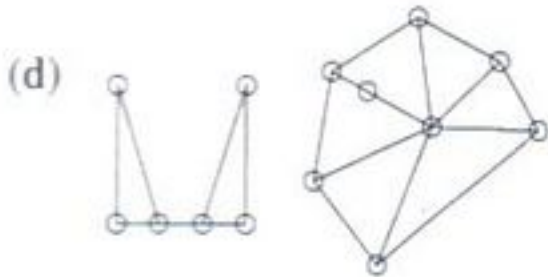
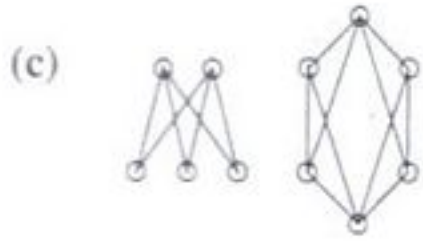
(b) Repeat the preceding questions for the graph on the right.

10. [BB] Verify Proposition 9.2.5 and Corollary 9.2.7 for the complete graph \mathcal{K}_n . What is the beta index of \mathcal{K}_n ? (See Exercise 5, Section 9.1.)
11. Verify Proposition 9.2.5 and Corollary 9.2.7 for the complete bipartite graph $\mathcal{K}_{m,n}$. What is the beta index of $\mathcal{K}_{m,n}$?
12. [BB] At most social functions, there is a lot of handshaking. Prove that the number of people who shake the hands of an odd number of people is always even.
13. Which of the graphs is a subgraph of the graph in Fig 9.19?

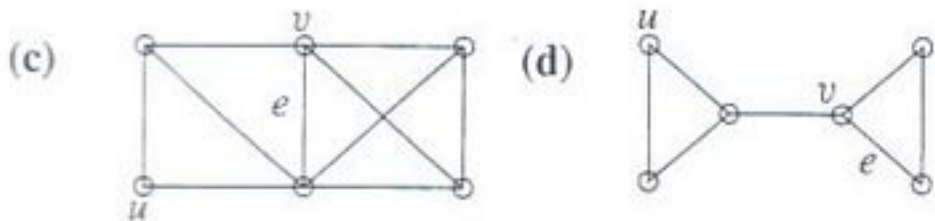
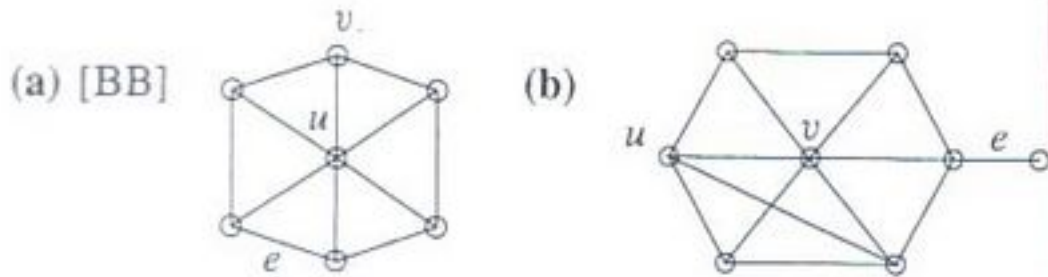


14. For each pair of graphs shown, discover whether or not the graph on the left is a subgraph of the one on the right. If it is not, explain why not. If it is, label the vertices of the subgraph, then use the same symbols to label the corresponding vertices of the graph on the right.

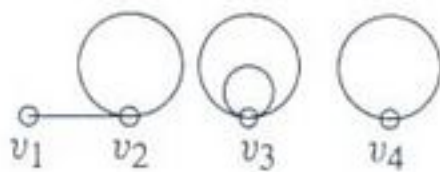




15. For each of the graphs shown, draw pictures of the subgraphs $G \setminus \{e\}$, $G \setminus \{v\}$, and $G \setminus \{u\}$.



16. (a) [BB] What are the degrees of the vertices in the pseudograph?



(b) [BB] Can there exist a graph with four vertices of degrees 1, 2, 3, and 4?

17. For each of the following sequences, determine if there exists a graph whose degree sequence is the one specified. In each case, either draw a graph, or explain why no graph exists.

- (a) [BB] 4, 4, 4, 3, 2
- (b) 100, 99, 98, ..., 3, 2, 2, 2
- (c) [BB] 5, 5, 4, 3, 2, 1
- (d) 1, 1, 1, 1, 1, 1
- (e) 5, 4, 3, 2, 1
- (f) 5, 4, 3, 2, 1, 1

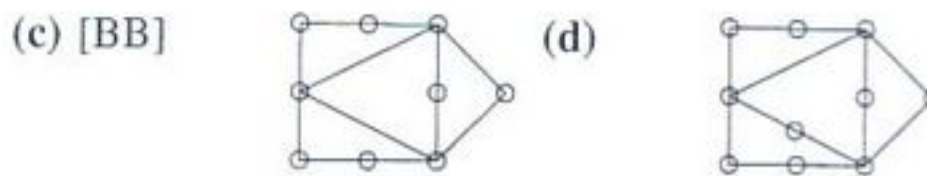
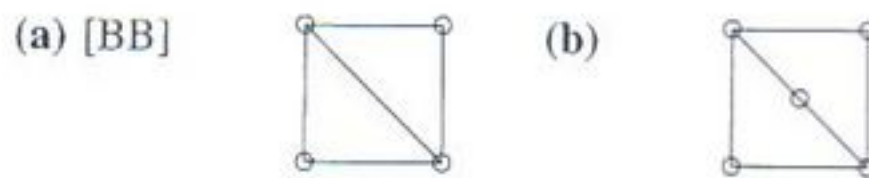
(g) 6, 6, 4, 2, 2, 2, 1, 1

18. Does there exist a graph with five vertices, every vertex incident with at least one edge but no two edges adjacent? Explain.

19. (a) [BB] A graph has five vertices of degree 4 and two vertices of degree 2. How many edges does it have?

(b) A graph has degree sequence 5, 5, 4, 4, 3, 3, 3, 3. How many edges does it have?

20. Determine whether or not each of the graphs is bipartite. In each case, give the bipartition sets or explain why the graph is not bipartite.



21. [BB] Prove that a graph which contains a triangle cannot be bipartite.

22. (a) Must a subgraph of a bipartite graph be bipartite?
 (b) Would your answer to (a) change if, in the definition of a bipartite graph, bipartition sets were required to be nonempty?

Explain your answers.

23. [BB] (Requires calculus) Prove that the number of edges in a bipartite graph with n vertices is at most $\frac{n^2}{4}$.

24. How many complete bipartite graphs have n vertices?

25. Let $V = \{1, 2, 3, \dots, n\}$.

(a) [BB] How many graphs are there with vertex set V ?

(b) How many of the graphs in (a) contain the triangle 123?

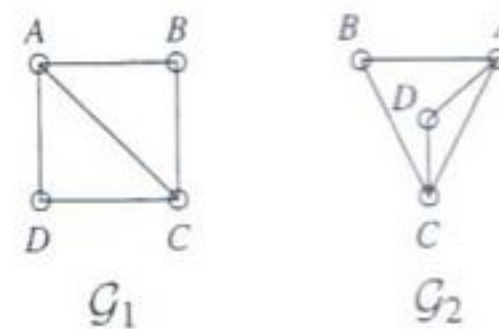
(c) [BB] What is the total number of triangles in all the graphs with vertex set V ?

(d) On average, how many triangles does a graph on n labeled vertices contain?

to have the same numbers of vertices and edges and the same degree sequences. See Exercises 7 and 8.

Answers to Pauses

6. To the right, we show one of several ways in which the vertices of G_2 can be labeled so that it becomes clear that the graph represented by each picture is the same.

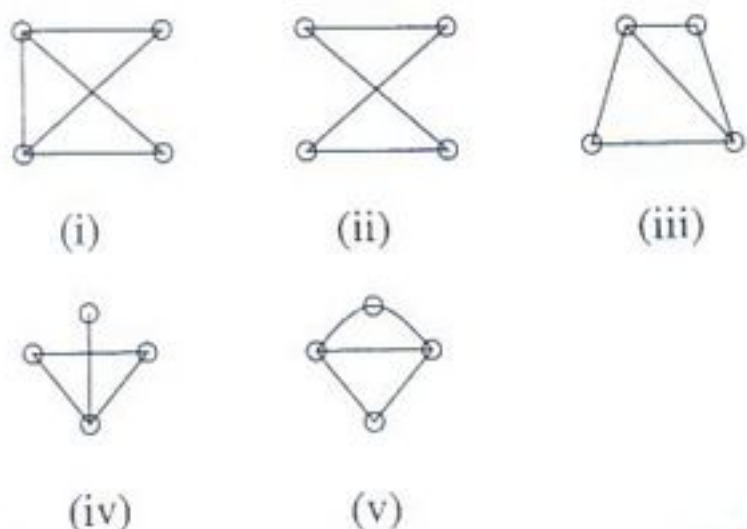


7. $\varphi(u) = B, \varphi(v) = C, \varphi(w) = A, \varphi(x) = D$. The isomorphism φ just relabels vertices of G_2 .

EXERCISES

The symbol [BB] means that an answer can be found in the Back of the Book.

1. [BB] For each of the ten pairs of graphs which can be obtained from those shown, either label the graphs so as to exhibit an isomorphism or explain why the graphs are not isomorphic.



2. (a) Draw a graph isomorphic to the one shown on the left, but with no crossover of edges.



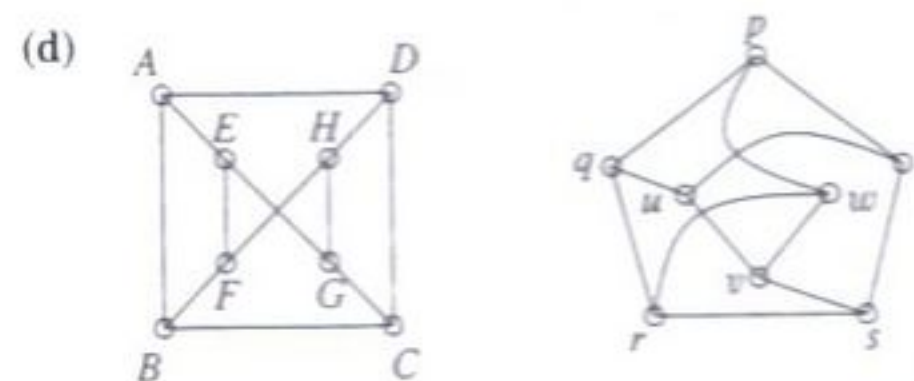
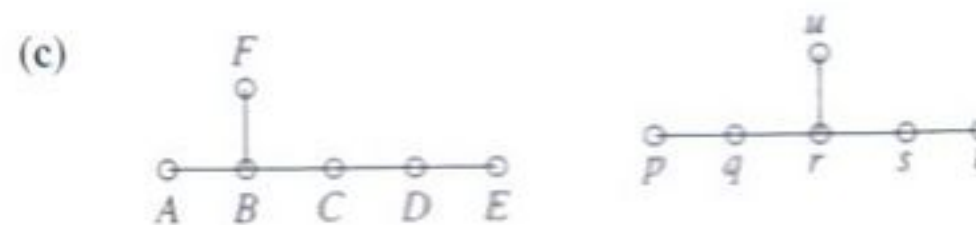
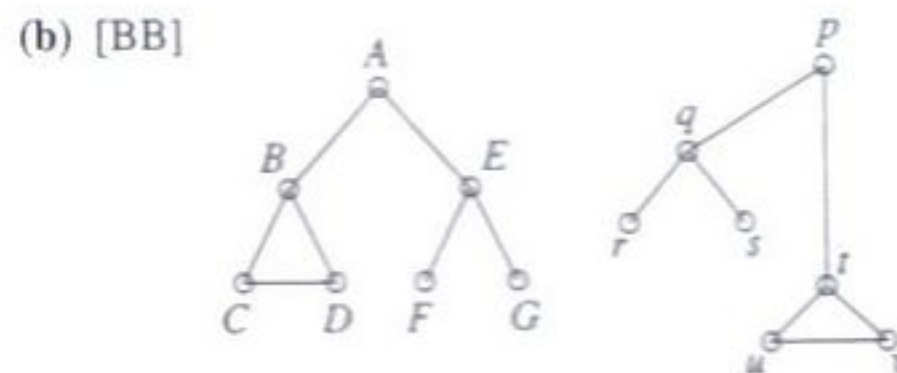
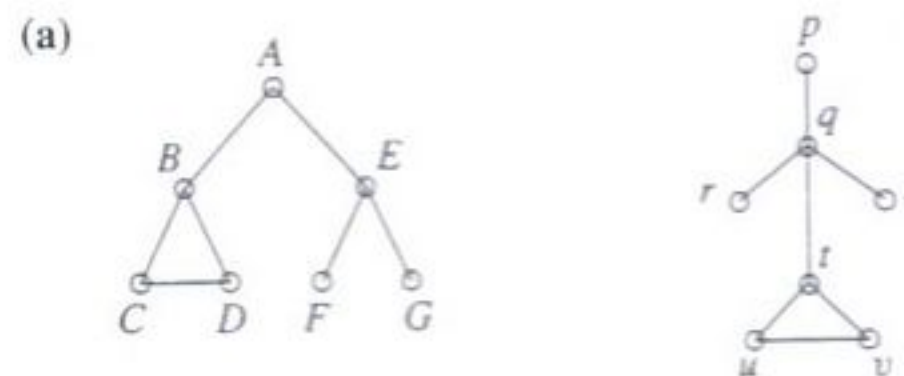
(b) Same as (a) for the graph on the right.

3. (a) [BB] Draw all nonisomorphic graphs on $n = 3$ vertices. Give the degree sequence of each.
 (b) Repeat part (a) for $n = 4$.

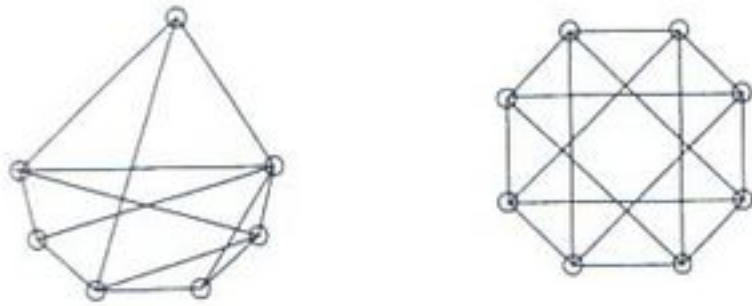
4. For each pair of graphs shown,

• if the graphs are not isomorphic, explain why not;

• if the graphs are isomorphic, exhibit an isomorphism from one to the other and relabel the graph on the right so as to show this isomorphism.



5. (a) [BB] Is the graph on the left isomorphic to $K_{3,4}$? Explain.



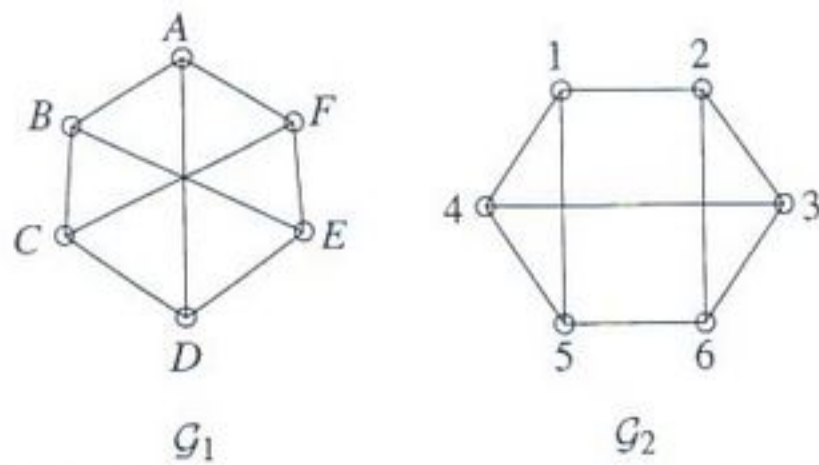
- (b) Is the graph on the right isomorphic to $K_{4,4}$? Explain.

6. [BB] Explain why any graph is isomorphic to a subgraph of some complete graph.

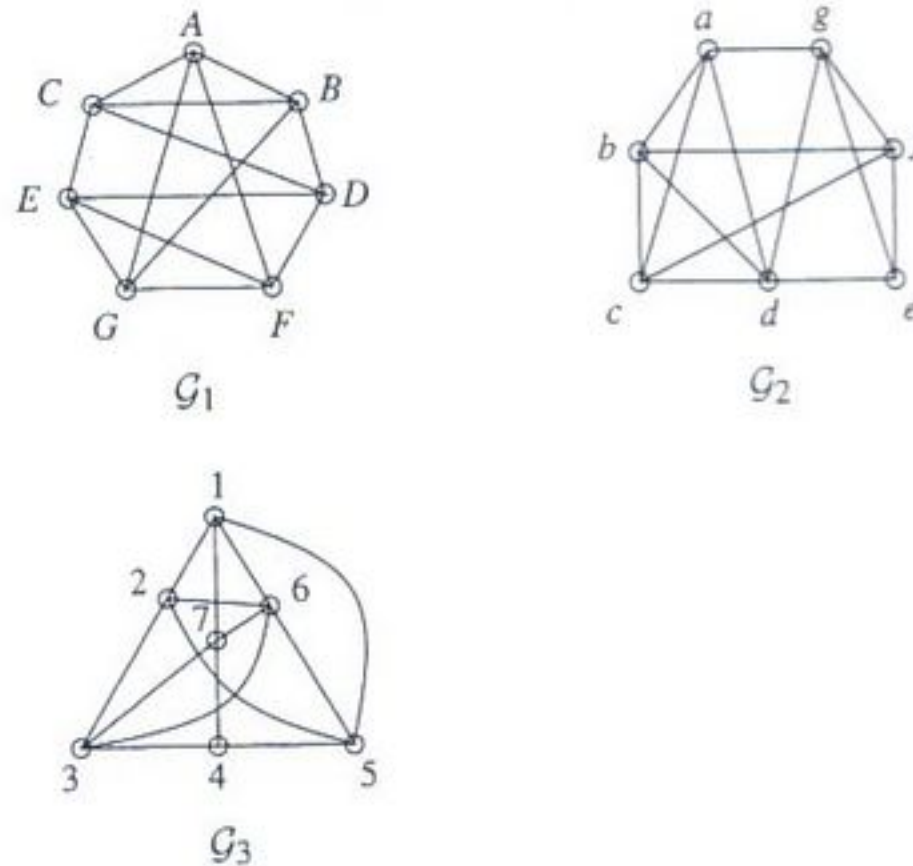
9.3.3 DEFINITION

Suppose v_1, \dots, v_n is a set of n vertices in a graph such that v_i and v_{i+1} are adjacent for $1 \leq i \leq n-1$ and v_n and v_1 are also adjacent. Then the set of these n vertices and the n edges $v_1v_2, v_2v_3, \dots, v_{n-1}v_n, v_nv_1$ is called an n -cycle. A 3-cycle is often called a *triangle* and a 4-cycle a *quadrilateral*.

7. (a) [BB] Prove that two graphs which are isomorphic must contain the same number of triangles.
 (b) Prove that, for any $n \geq 4$, two isomorphic graphs must contain the same number of n -cycles.
 (c) How many edges are there in the graphs G_1 and G_2 ? How many vertices? What is the degree sequence of each graph? Are the graphs isomorphic? Explain.

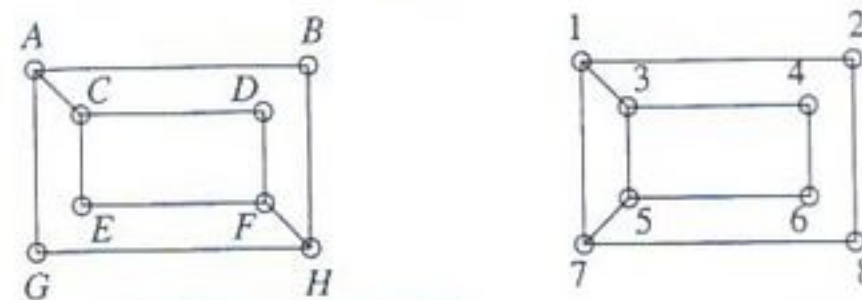


8. Consider the following three graphs.



- (a) How many vertices and how many edges are there in each graph? What is the degree sequence of each graph? How many triangles are in each graph?
 (b) For each pair of graphs, either exhibit an isomorphism between vertex sets or explain why the graphs are not isomorphic.

9. Show that the following graphs are not isomorphic.



10. (a) [BB] Suppose that graphs G and H have the same numbers of vertices and the same numbers of edges, and suppose that the degree of every vertex in G and in H is 2. Are G and H necessarily isomorphic? Explain.
 (b) Suppose that graphs G and H have the same number of vertices and the same number of edges. Suppose that the degree sequences of G and H are the same and that neither graph contains a triangle. Are G and H necessarily isomorphic? Explain.

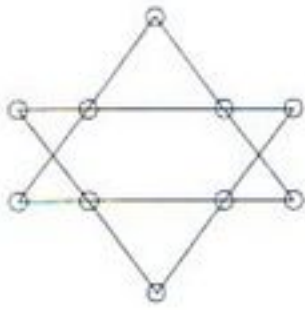
REVIEW EXERCISES FOR CHAPTER 9

1. In the Königsberg Bridge Problem, a tragic fire destroys the bridge from B to C and also one of the bridges from A to D . (See Fig 9.1.) Draw a graph representing the new situation. Show that it is now possible for someone to start on land mass B and walk over each of the bridges exactly once, returning to land mass B again.
2. (a) Draw a configuration of four houses and two utilities, each house connected to each utility, but with no crossovers.
 (b) Let n be any positive integer. Motivated by 2(a), suggest a general result concerning n houses and 2 utilities. Draw a graph supporting your answer.

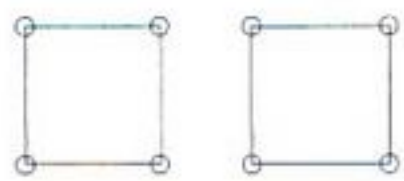
You need to label the vertices to be able to write down the circuit. "pseudograph" because loop and multiple edges allowed.

4. In each of the pseudographs shown, either describe an Eulerian circuit by numbering the edges or explain why no Eulerian circuit exists.

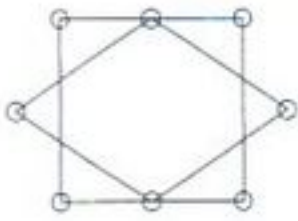
(a) [BB]



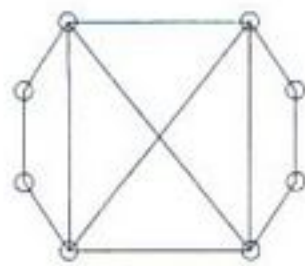
(b)



(c)



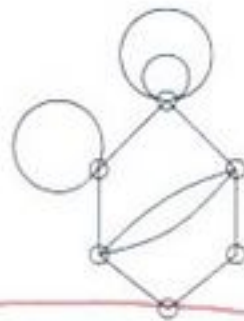
(d)



(e)

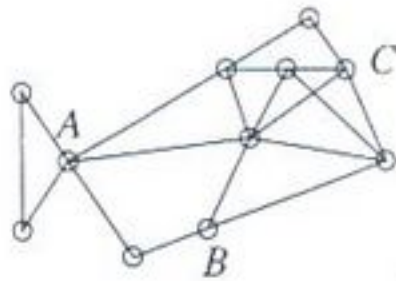


(f)



5. Suppose we modify the definition of Eulerian circuit by omitting the reference to vertices. Thus we propose that an Eulerian circuit be a circuit which contains every edge of a graph. Does Theorem 10.1.4 remain true? Explain.

6. (a) [BB] Is there an Eulerian trail from A to B? If yes, find one; if not, explain why not.



(b) Same question for A to C.

7. [BB] (Fictitious) A recently discovered map of the old town of Königsberg shows that there was a ferry operating between the areas labeled C and D in Fig 9.1.

(a) Is it possible to start on some land area, cross over each bridge exactly once, take the ferry exactly

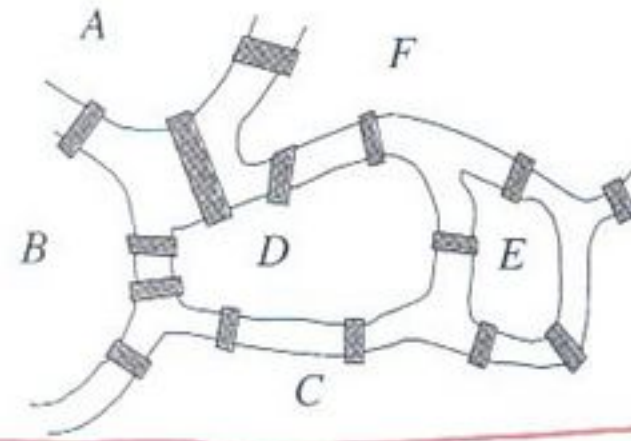
once, and return to the starting point? Explain your answer.

(b) Is it possible to start on some land mass, walk over each bridge exactly once, take the ferry exactly once, and finish on some land mass (possibly different from the starting point)? Explain.

8. Euler's original article about the Königsberg Bridge Problem, which is dated 1736, presents a second similar problem with two islands, four rivers flowing around them, and 15 bridges connecting various land masses, as shown below.

(a) Is it possible to tour the region starting and finishing in the same area having walked over every bridge exactly once? Either describe such a tour or explain why none is possible.

(b) Is it possible to tour the region (with perhaps different starting and stopping points) having walked over every bridge exactly once? Either describe such a tour or explain why none is possible.



9. [BB] In Exercise 10 of Section 9.3, you were asked whether graphs \mathcal{G} and \mathcal{H} with the same numbers of vertices and edges and with every vertex in each graph of degree 2 need be isomorphic. Answer this question again, assuming in addition that the graphs are connected.

10. Suppose \mathcal{G}_1 and \mathcal{G}_2 are Eulerian graphs with no vertices in common. Let v_1 be a vertex in \mathcal{G}_1 and let v_2 be a vertex in \mathcal{G}_2 . Join v_1 and v_2 with a single edge. What can be said about the resulting graph and why?

11. (a) [BB] For which values of $n > 1$, if any, is \mathcal{K}_n Eulerian?

(b) [BB] For which values of $n > 1$, if any, does \mathcal{K}_n possess an Eulerian trail? Explain.

12. (a) Find a necessary and sufficient condition on natural numbers m and n in order for $\mathcal{K}_{m,n}$ to be Eulerian. Prove your answer.