Study Problems for Test III with Solutions

Dr. Holmes

April 1, 2004

This document is certainly much longer than the test. Nor is there any guarantee that the coverage is exactly the same as that on the test. Some of the individual problems are harder than any individual test question is likely to be, but they should help you to think about the right things.

This document does not give complete coverage for chapter 10, though it does have some chapter 10 questions: for full chapter 10 coverage, also look at assignment XII.

The test starts with the construction of the integers in chapter 7 and continues through section 10.5.

Solutions will be posted on the web on Wednesday afternoon or Thursday.

1. Prove, using the representation of the integers in chapter 7, that the square of any integer is either zero or positive. By "a is positive", I mean $a > 0$. Your proof should reduce entirely to calculations with natural numbers only, plus applications of the definitions of $>$ and multiplication on the represented integers.

   Solution: An integer is written as an equivalence class $[(m, n)]$ for some natural numbers m and n. Using the definition of multiplication, we find that $[(m, n)] * [(m, n)] = [m^2 + n^2, 2mn]$

   $[(1, 1)]$ is the integer 0.

   $[m^2 + n^2, 2mn] \geq [(1, 1)]$ means $m^2 + n^2 + 1 \geq 2mn + 1$ or equivalently $m^2 + n^2 > 2mn$. (This uses the definition of order for represented integers).

   Suppose $m > n$.

Then we have $(m-n)^2 = m^2 - 2mn + n^2$ a natural number, so $m^2 + n^2$ must be greater than $2mn$.

Suppose $n > m$. Then we have $(n-m)^2 = n^2 - 2mn + m^2$ a natural number, and once again $m^2 + n^2 > 2mn$.

If $m = n$, it follows that [(m,n)] is the integer zero and its square is zero.

This is actually quite tricky (it is hard to see what you are allowed to do). Don't expect anything this hard.

2. Convert $153_{\texttt{ten}}$ to base 8.

   Solution 153 divided by 8 is 19 remainder 1

   19 divided by 8 is 2, remainder 3.

   2 divided by 8 is 0, remainder 3.

   The answer is 231_8.

   Convert 234_8 to base 10.

   Solution $2 * 8^2 + 3 * 8 + 4 = 156$

3. Give the complete addition and multiplication tables for mod 4 arithmetic.

+	0	1	2	3
0	0	1	2	3
1	1	2	3	0
2	2	3	0	1
3	3	0	1	2

*	0	1	2	3
0	0	0	0	0
1	0	1	2	3
2	0	2	0	2
3	0	3	2	1

Give the complete addition and multiplication tables for base 4.

+	0	1	2	3
0	0	1	2	3
1	1	2	3	10
2	2	3	10	11
3	3	10	11	12

*	0	1	2	3
0	0	0	0	0
1	0	1	2	3
2	0	2	10	12
3	0	3	12	21

Notice that these are not the same thing!

Convert 27 and 45 to base 4. Carry out the calculations 27+45 and (27)(45) in base 4. Convert your results back to base 10 and check that you have the right answers.

$27 = 16+8+3 = 123_4$

$45 = 32+12+1 = 231_4$

adding:

$$\begin{array}{cccc} & 1 & 1 & \\ & 1 & 2 & 3 \\ & 2 & 3 & 1 \\ \hline 1 & 0 & 2 & 0 \end{array}$$

and you can check that $24+45=72 = 1020_4$.

$$\begin{array}{cccccc} & & & 1 & 2 & 3 \\ & & & 2 & 3 & 1 \\ \hline & & & 1 & 2 & 3 \\ & 1 & 1^2 & 0^2 & 1 & \\ & 3^1 & 1^1 & 2 & & \\ \hline 1 & 0 & 2 & 3 & 3 & 3 \end{array}$$

and $102333_4 = 1215_{\texttt{ten}} = (27_{\texttt{ten}}) * (45_{\texttt{ten}})$.

4. Compute $\texttt{gcd}(6105, 21390)$.

   21390 divided by 6105 is 3 remainder 3075.

   6105 divided by 3075 is 1, remainder 3030.

   3075 divided by 3030 is 1, remainder 45.

   3030 divided by 45 is 67, remainder 15.

   45 divided by 15 is 3 remainder 0.

   So the gcd is 15.

   Determine integers m and n such that $6105m+21390n = \texttt{gcd}(6105, 21390)$.

   21390 - (3)(6105) = 3075.

   3030 = 6105 - 3075 = 6105 - (21390 - (3)(6105)) = (4)(6105) - 21390

   45 = 3075 - 3030 = (21390 - (3)(6105)) - ((4)(6105) - 21390) = (2)(21390) - (7)(6105)

   15 = 3030 - (67)(45) = ((4)(6105) - 21390) - (67)((2)(21390) - (7)(6105))
   = (4 + (67(7))(6105) - (1 + (67(2)))(21390) = (473)(6105) - (135)(21390)

   So gcd(21390,6105) = 15 = 473*6105 - 135*21390.

   Find integers m and n such that $6105m + 21390n = 20$, if possible. If it is not possible, explain why not.

   It can't be done, since 15 doesn't go evenly into 20.

   Find integers m and n such that $6105m + 21390n = 30$, if possible. If it is not possible, explain why not.

   30 = 2*15 = (2*473)6105 - (2*135)21390 = 946*6105 - 270*21390.

5. There is a theorem which says that if p is a prime and $p|abc$, then either $p|a$, $p|b$, or $p|c$. Show that this is not necessarily true if p is not a prime: choose values of p, a, b, c such that $p|abc$ and none of the statements $p|a$, $p|b$, and $p|c$ are true.

   For example, consider 30, which goes evenly into (4)(9)(25) = 900 but doesn't go evenly into either 4, 9, or 25; the idea is that the prime factors of 30 (2, 3, 5) each go evenly into different factors in the product.

6. Give an argument based on prime factorizations to show that no fraction $\frac{a}{b}$ with a and b positive integers can be a square root of 3.

   Hint: a^2 and b^2 must each have an even number of factors of 3 in their prime factorizations (why?). Now think about cancelling factors of 3 in $\frac{a^2}{b^2}$.

   Since the number of factors of 3 in the numerator a^2 and the number of factors of three in the denominator b^2 are both even (twice the number of factors in a and twice the number of factors in b), the number of factors of 3 in the numerator of the simplified fraction $\frac{a^2}{b^2}$ must also be even, which is not possible if the quotient is to be equal to $3 = \frac{3}{1}$. So there can be no a and b in the natural numbers such that $\frac{a^2}{b^2} = 3$.

7. If $c = ab$, and $a \geq \sqrt{c}$, then $b \leq \sqrt{c}$ must be true. Use this fact to show that if c has no prime factor $\leq \sqrt{c}$, then it must be a prime number itself. Carefully explain why this is true (it does take a little bit of care).

   The argument (proving the contrapositive) should begin: "suppose that c is not a prime number. Then $c = ab$, with neither a not b equal to c or 1 ..."

   Solution: If c is not a prime then $c = ab$ for some a and b, neither of which is equal to 1. Suppose a is the smaller of the two numbers; it cannot be greater than $\sqrt{c}$, or our fact would make $b < \sqrt{c}$ and so a would not be the smaller of the two. a may not be a prime itself, but because it is not equal to 1 it has a prime factor, and this prime factor is $\leq a$ and so less than or equal to $\sqrt{c}$.

   If I want to check whether 1000001 is a prime, what is an upper limit on how large the smallest prime factor of 1000001 could be?

   The smallest prime factor can be no greater than $\sqrt{1000001} < 1001$.

8. Show that the definition of multiplication for our representation of the rationals is independent of representatives. (You should know what this means; I'm not providing any hints here, but you can look at the similar problem in assignment XI if you want to see the hints I gave there. I will tell you that your calculations should reduce to calculations with integers alone).

Suppose $\frac{a}{b} = \frac{a'}{b'}$ and $\frac{c}{d} = \frac{c'}{d'}$; what we need to show is that $(\frac{a}{b})(\frac{c}{d})$ is equal to $(\frac{a}{\prime}b')(\frac{c}{\prime}d')$.

So assume $\frac{a}{b} = \frac{a'}{b'}$ and $\frac{c}{d} = \frac{c'}{d'}$; this is the same as assuming $ab' = a'b$ and $cd' = c'd$.

What we need to prove is $(\frac{a}{b})(\frac{c}{d}) = (\frac{a}{\prime}b')(\frac{c}{\prime}d')$, which is equivalent to $\frac{ac}{bd} = \frac{a'c'}{b'd'}$ by the definition of multiplication of fractions, which is in turn equivalent to $acb'd' = a'c'bd$: so $acb'd' = a'c'bd$ is what we need to show.

$acb'd' = (ab')cd' = (a'b)(c'd) = a'c'bd$: the first equation follows from regrouping properties of multiplication of integers, the second equation follows from the hypotheses $ab' = a'b$ and $cd' = c'd$, and the third follows from regrouping again.

What I write $\frac{a}{b}$ can also be written $[(a, b)]$, of course, but we did define $\frac{a}{b}$ as $[(a, b)]$, which gives us a chance to use familiar notation.

9. Show that if $\frac{a}{b} < \frac{c}{d}$, then $\frac{a}{b} < \frac{ad+2bc}{3bd} < \frac{c}{d}$. If I draw $\frac{a}{b}$ and $\frac{c}{d}$ on a number line, where is $\frac{ad+2bc}{3bd}$? (This is trickier than any test question is likely to be (so don't spend too much time on it if you don't see how to do it), but it is a good question for thinking about density).

   Solution: Don't worry about this one!

10. Show that for any rational numbers a and $b \neq 0$ (you might want to write them as fractions) $a + b\sqrt{2}$ is irrational. (Hint: if you set this equal to a fraction, you can solve for $\sqrt{2}$).

    If $a + b\sqrt{2} = c$, where c is a rational number, then $\sqrt{2} = \frac{(}{c} - a)b$, which will be rational because a, b, c are rational, which is impossible.

    Show that for any $a < b$, $a + \frac{\sqrt{2}}{2}(b - a)$ is greater than a and less than b.

    Since $0 < \frac{\sqrt{2}}{2} < 1$, we have $a = a + 0(b - a) < a + (\frac{\sqrt{2}}{2})(b - a) < a + 1(b - a) = b$; this follows by adding a to each side of the inequalities then multiplying them by the positive quantity $b - a$.

    Now use these results to show that for any rational numbers a and b, there is an irrational number r such that $a < r < b$.

The number $a + \frac{\sqrt{2}}{2}(b-a) = a + (\frac{1}{2})(b-a)\sqrt{2}$ is irrational by the first fact, and lies between a and b by the second fact.

This is too complex again to be a test question, but there are ingredients in it that might go into a test question.

11. Compute the repeating decimal representation of $\frac{3}{11}$ in base 10.

    This will be .272727...

    What are the two distinct decimal representations of $\frac{1}{8}$ in base 10? (There really are two!)

    .125 and 1249999....

    Present the repeating decimal 0.2313131... as a fraction. Show a calculation justifying your conclusion (admittedly this will be somewhat informal!)

    $(99)(.2313131\ldots) = (100)(.2313131\ldots) - .2313131\ldots = 23.131313\ldots - .2313131\ldots = 23.1 - .2 = 22.9$ so $0.2313131\ldots = \frac{22.9}{99} = \frac{229}{990}$.

    Explain why the infinite decimal 0.101001000100001... represents an irrational number.

    It neither terminates nor ends in an infinitely repeating block of digits.

12. Compute the repeating decimal representation of $\frac{1}{3}$ in base 7.

    This will be 0.2222222...

    What is the value of the repeating base 3 "decimal" 0.1111...? Write it as a fraction.

    Its value is $\frac{1}{2}$.

13. Give a clear description (a diagram is acceptable) of a bijection between the set of natural numbers and the set of ordered pairs of natural numbers. Show the ordered pairs associated with the numbers from one to ten explicitly (list them), and make it clear using a diagram or verbal description how the process would continue.

    Look at the picture in the book illustrating the theorem that the set of rationals is countable.

14. Give a counterexample showing that it is not always true that $\phi(ab) = \phi(a)\phi(b)$ (where ϕ is the Euler phi function defined in section 10.3). This is straightforward (compute several values of ϕ and you should find a counterexample).

    $\phi(2) = 1$, $\phi(4) = 2$, and $\phi(8) = 4 \neq (1)(2)$.

    Harder (certainly not a test question): look for a special condition on a and b which makes the equation $\phi(ab) = \phi(a)\phi(b)$ true. This is just something to think about if you like math...

15. A license plate contains three letters and four digits.

    How many possible license plates are there satisfying each of the following sets of conditions?

    (a) no extra conditions (all possible plates)
    $26^3 * 10^4$

    (b) no letter or digit appears more than once (no repetitions at all)
    $26 * 25 * 24 * 10 * 9 * 8 * 7$

    (c) no letter or digit is immediately followed by itself.
    $26 * 25 * 25 * 10 * 9 * 9 * 9$

    (d) some letter is repeated *and/or* some digit is repeated (easy if you did one of the earlier parts).
    $26^3 * 10^4 - 26 * 25 * 24 * 10 * 9 * 8 * 7$ (take all license plates and subtract the ones with no repetitions)

    (e) some letter is repeated *and* some digit is repeated (be careful!)
    $(26^3 - 26 * 25 * 24) * (10^4 - 10 * 9 * 8 * 7)$ – the idea is to compute how many sequences of three letters with at least one repetition there are then how many sequences of four numbers with at least one repetition, then multiply these two quantities.

    (f) some letter is immediately followed by itself *and* some digit is immediately followed by itself
    $(26^3 - 26 * 25 * 25) * (10^4 - 10 * 9 * 9)$

    (g) There is at least one A and at least one 1. Repetitions are allowed.
    $(26^3 - 25^3) * (10^4 - 9^4)$ will do it.

(h) There is exactly one A and exactly one 1. There is no other restriction on repetitions.

$3 * 25^2 * 4 * 9^3$ three ways to choose the place where the A is, then two choices of a non-A letter, then four ways to choose the place where the 1 is and three choices of a non-1 digit.