Math 187 Sample Test II Questions

Dr. Holmes

February 25, 2004

This version contains the missing proof that compositions of injections are injections from the in-class version, corrects the typo in the Lucas number problem's solution, but does not contain the handwritten answers found in the class handout.

I solved the last two problems in class.

1. Prove that for any natural numbers x, y, z, if $x < y$ then $xz < yz$, using the definition of $<$ and the axioms from chapter 4 (which would be given).

   Solution:

   Let x, y, z be arbitrarily chosen natural numbers.

   Suppose $x < y$. This means that there is a natural number k such that $x + k = y$.

   We want to show $xz < yz$. Notice that

   $$yz =$$

   by substitution

   $$(x + k)z =$$

   by commutativity of mult.

   $$z(x + k) =$$

   by distributivity

$$zx + zk =$$

by commutativity of mult. (twice)

$$xz + xk$$

Now $xz < xz + xk$ by definition of $<$, so $xz < yz$ by substitution of equals for equals.

This completes the proof.

2. Prove that for any x, y, z natural numbers with $y > z$, $x(y - z) = xy - xz$, using the axioms and the definition of subtraction (which would be given).

   Recall that $x - y$ is defined as the number k such that $y + k = x$; there is at most one such number by the cancellation property of addition.

   If $y < x$ then $x - y$ exists (as a natural number) by the definition of $<$.

   Now we begin the proof.

   Let x, y, z be natural numbers.

   Suppose that $y > z$. Then by definition of $>$ and $<$, there is k such that $z + k = y$, and by the definition of subtraction $k = y - z$.

   We want to show that $x(y - z) = xy - xz$ This is equivalent to $x(y - z) + xz = xy$.

   Now
   $$x(y - z) + xz =$$
   by definition of k above
   $$xk + xz =$$
   by commutativity of addition
   $$xz + xk =$$
   by dist.
   $$x(z + k) =$$
   by choice of k above
   $$xy$$

so, since $x(y-z)+xz = xy$, we have $xy-xz = x(y-z)$ by the definition of subtraction, which is what we want.

3. Prove using mathematical induction that

$$(\Sigma_{i=1}^{n} i) = \frac{n(n+1)}{2}$$

This proof appears in the Feb. 3 notes.

4. The *Lucas numbers* are defined by the recursive definition

$$u_1 = 1; u_2 = 3; u_{n+2} = u_n + u_{n+1}$$

Prove by induction that $u_{n+2} = f_n + 3f_{n+1}$ for each natural number n, where f_n refers to the nth Fibonacci number.

We have to use the special form of induction where one proves $P(1)$ and $P(2)$ as the basis step and shows that if $P(n)$ and $P(n+1)$ are true, so is $P(n+2)$.

Basis step: $u_3 = f_1+3f_2$: $u_3 = u_1+u_2 = 1+3 = 4 = 1+3(1) = f_1+3f_2$; $u_4 = f_2 + 3f_3$: $u_4 = u_2 + u_3 = 3 + 4 = 7 = 1 + 3(2) = f_2 + 3f_3$

Induction step: Show that if $u_{k+2} = f_k+3f_{k+1}$ and $u_{k+3} = f_{k+1}+3f_{k+2}$ then $u_{k+4} = f_{k+2} + 3f_{k+3}$

Suppose that $u_{k+2} = f_k + 3f_{k+1}$ (this is the inductive hypothesis).

$$u_{k+4} =$$

by recursive definition of Lucas numbers

$$u_{k+2} + u_{k+3} =$$

by ind. hyp. (using both parts)

$$(f_k + 3f_{k+1}) + (f_{k+1} + 3f_{k+2}) =$$

by algebra

$$(f_k + f_{k+1}) + 3(f_{k+1} + f_{k+2}) =$$

by recursive definition of Fibonacci numbers

$$f_{k+2} + 3f_{k+3}$$

so $u_{k+4} = f_{k+2} + 3f_{k+3}$, which is what we needed to show.

5. List all functions from the set $\{a, b, c\}$ to $\{1, 2\}$ (you may write them as sets of ordered pairs or illustrate them with arrow diagrams). Identify any of these functions which may be injections. Identify any of these functions which may be surjections.

   On handwritten sheet.

6. Determine how many injections there are from $\{1, 2, 3\}$ into $\{a, b, c, d\}$; draw a tree diagram to illustrate your answer (you may choose not to draw the entire tree, but just enough to make it clear how many branches it has).

   On handwritten sheet.

7. Look at the diagrams provided. Determine which of the following expressions involving compositions and inverses are meaningful and write down the ones which are meaningful as sets of ordered pairs.

   On handwritten sheet.

   (a) fg
   (b) gf
   (c) h^{-1}
   (d) g^{-1}
   (e) $h^{-1}g$

8. Prove that if f and g are surjections, so is the composition gf. Prove the same result for injections.

   surjections:

   Suppose that $f : A \to B$ and $g : B \to C$ are surjections. We need to show that gf is a surjection from A to C. This is equivalent to the assertion that for any z in C, there is x in A such that $g(f(x)) = z$ (This is not something we know, but something we need to show!)

   Let z be any element of C. Because g is a surjection from B to C, there is y in B such that $g(y) = z$. Because f is a surjection from A to B, there is x in A such that $f(x) = y$. Since $f(x) = y$ and $g(y) = z$, substitution of equals for equals gives $g(f(x)) = z$, equivalently $gf(x) = z$, so gf is surjective, since z was a completely arbitrary element of C.

injections:

Suppose that $f : A \to B$ and $g : B \to C$ are injections. We need to show that gf is an injection from A to C. This is equivalent to showing that for any x, y in A, if $g(f(x)) = g(f(y))$, then $x = y$.(Once again, this is not something we know, but something we need to show!)

Let x and y be elements of A. Suppose that $g(f(x)) = g(f(y))$. Because f is an injection, we know that if $g(a) = g(b)$, for any a, b in B, we must have $a = b$. Letting $a = f(x)$ and $b = f(y)$, we see that $g(f(x)) = g(f(y))$ implies $f(x) = f(y)$, so $f(x) = f(y)$ follows from our assumption. Because f is an injection, we know that for any x, y in A if $f(x) = f(y)$ then $x = y$. Since we showed $f(x) = f(y)$ already, we have shown that $x = y$, which is what we needed to complete the proof.

9. Argue using the Pigeonhole Principle that from any four numbers a, b, c and d, one can choose two numbers whose difference is divisible by three.

   The difference between two numbers (here we should think integers) is divisible by three just in case the two numbers have the same remainder on division by three. There are three possible remainders, 0,1,2. If we associate each of four numbers with one of the values 0,1,2, two of the numbers need to be associated with the same value by the pigeonhole principle: so two of the four numbers must have the same remainder on division by three and so must have the difference between them divisible by three.

10. Prove that the set of natural numbers all of whose digits in base ten are different is finite. Hint: this also uses the Pigeonhole Principle.

    If all the digits in a natural number are different, it can have no more than ten digits (this is the application of the Pigeonhole Principle).

    Any number with ten or fewer digits is less than or equal to ten billion ($10^1 0$). So our set is a subset of the finite set $\mathcal{N}_{10^1 0}$, and so is finite.

11. Define the function $f(n)$ recursively: $f(1) = 13$; if $f(n)$ is even,

$$f(n+1) = \frac{f(n)}{2};$$

if $f(n)$ is odd,

$$f(n+1) = 3n + 1.$$

Compute enough values to show that f is neither an injection nor a surjection. You must explain correctly why f is not an injection and why f is not a surjection.

I'll do this one in class; it is like an example we did earlier.

12. Prove that the relation on natural numbers defined by "xy is a perfect square" is an equivalence relation. Partition the numbers ≤ 20 into equivalence classes under this relation.

    I'll do this one in class.