Math 187 Test III, Spring 2013

Dr. Holmes

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The exam will begin at 9 am and end officially at 10:15 am; what will actually happen is that you will get a five minute warning to finish at 10:15 am. You are allowed your test paper, your writing instrument, and a non-graphing calculator.

In all of the problems where I tell you to do one of the two parts, and that if you do both parts your best work will count, doing very well on both parts will give a chance of extra credit. But do not attempt extra credit opportunities until you have written the entire exam.

1. Functions

   (a) Three sets of ordered pairs are given. One is not a function (explain). One is a function which is not one-to-one (explain). One is a function with an inverse (compute the inverse).

$$\{(1,1),(2,2),(3,1),(4,3)\}$$

   a function, no inverse because 1 and 3 are both sent to 1.

$$\{(0,1),(0,2),(1,3),(1,4)\}$$

   not afunction, 0 is sent to both 1 and 2.

$$\{(1,1),(2,3),(3,4),(4,2)\}$$

   a function with inverse $\{(1,1),(3.2),(4,3),(2,4)\}$

   (b) How many functions from $\{1,2,3\}$ to $\{a,b,c,d\}$ are there? How many of them are one to one? How many of them are onto $\{a,b,c,d\}$?

There are 4^3 such functions (32) of which $(4)(3)(2) = 24$ are one to one. None of them are onto.

2. Function composition

   For each pair of functions, compute $f \circ g$ and $g \circ f$, or indicate why they are undefined. In each part, label your answers clearly so that I can tell which is $f \circ g$ and which is $g \circ f$.

   (a) $f = \{(1,2),(2,3),(3,4),(4,5)\}$; $g = \{(2,0),(3,1),(4,2),(5,3)\}$. typo in this question corrected during the exam

   $f \circ g$ is undefined because $f(g(2))$ would be $f(0)$ which is not defined.

   $g \circ f$ is $\{(1,0),(2,1),(3,2),(4,3)\}$ (both would be undefined with the original typo).

   (b) $f(x) = \sqrt{x}$; $g(x) = x^2 + 1$

   $f \circ g$: $\sqrt{x^2+1}$

   $g \circ f$: $(\sqrt{x})^2 + 1$ ($x + 1$ with restricted domain)

3. Recurrence relations. Do one of the two parts. If you do both, you will receive credit for your best work.

   (a) The sequence a_n is defined by $a_n = 4a_{n-1} - 3a_{n-2}$ for each $n \geq 2$, $a_0 = 2$, $a_1 = 4$.

   Compute the next four terms of this sequence.

   Derive a formula for the nth term which does not involve a recurrence relation.

   find x such that $x^2 - 4x + 3 = 0$: $x = 1$ or 3.

   so $a_n = A1^n + B3^n$

   plugging in $n = 0$ and $n = 1$ gives

   $A + B = 2$ $A + 3B = 4$

   from which $B = 1$, $A = 1$ so $a_n = 1 + 3^n$

   and the values you compute for the sequence should agree with this:

   2,4,(4)(4)-(3)(2) = 10, (4)(10)-(3)(4) = 28, and so forth.

(b) Find a polynomial formula for the sequence whose first few terms are

$2, 1, 2, 5, 10, 17, 26, \ldots$ by the method of differences illustrated in the book. Write your formula first in the form

$$A\binom{n}{0} + B\binom{n}{1} + C\binom{n}{2} + \ldots$$

and then in polynomial form

$$a + bn + cn^2 + \ldots$$

method of differences gives

2 1 2 5 10

-1 1 3 5

2 2 2

0 0

so $a_n = 2(\frac{n(n-1)}{2} - n + 2 = n(n-1) - n + 2 = n^2 - 2n + 2$

4. Permutations

(a) Write the permutation $\begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 4 & 1 & 5 & 6 & 2 \end{bmatrix}$ in disjoint cycle notation.

(b) Compute the composition $\sigma \circ \tau$ where $\sigma = (1356)(24)$ and $\tau = (125)(364)$. (another typo in original corrected during the exam)
$(1453)(26)$

(c) Express the permutation $(1345)(267)$ as a product of transpositions. Is it an even or odd permutation?
$(15) \circ (14) \circ (13) \circ (27) \circ (26)$. it is a composition of five transpositions, so it is odd.

5. Counting problems. Do one of the two problems. If you do both, you will receive credit for your best work.

   (a) How many natural numbers less than or equal to 120 are divisible by either 2, 3 or 5?

   There is a mistake in the phrasing of this question, which most people didnt notice. I should have said positive integers not natural numbers. The correct answer to the question as written is one larger (adding in zero). I accepted the answer I give below as correct.

   60 divisible by 2

   40 divisible by 3

   24 divisible by 5

   20 divisible by both 2 and 3

   12 divisible by both 2 and 5

   8 divisible by both 3 and 5

   4 divisible by all three

   Use inclusion/exclusion

   $60 + 40 + 24 - 20 - 12 - 8 + 4 = 88$ This answer was accepted for full credit.

   The correct answer would increase each of the 7 numbers computed initially by 1, and the final answer would be 89. No one gave this answer; one student noticed that the first 7 numbers needed to be one larger, but made a different mistake.

(b) How many nine-digit numbers with all digits nonzero have exactly three 1's?

The answer is 9-choose-3 times 8^6. Choose three places to put the ones then place one of the 8 nonzero digits other than 1 in each of the six remaining positions.

6. Prove by mathematical induction that the sum of the first n positive integers, which can be written $\sum_{i=1}^{n} i$ or $1 + 2 + 3 + \ldots + n$ is equal to $\frac{n(n+1)}{2}$. Pay careful attention to the structure of your proof, clearly indicating what the basis step is, what the induction hypothesis is, what the induction goal is, and where you use the induction hypothesis in your proof.

   basis, 1=1.

   induction, assume that $1 + \ldots + k = \frac{k(k+1)}{2}$. The goal is to show $1 + \ldots + k + (k+1) = \frac{(k+1)((k+1)+1)}{2}$

   add $k+1$ to both sides. $1 + \ldots + k + (k+1) = \frac{k(k+1)}{2} + (k+1)$

   Algebra shows that $\frac{k(k+1)}{2} + (k+1)$ and $\frac{(k+1)((k+1)+1)}{2}$ simplify to the same thing.

7. Prove by mathematical induction that $n^3 + 5n$ is divisible by 3 for any n. Pay careful attention to the structure of your proof, clearly indicating what the basis step is, what the induction hypothesis is, what the induction goal is, and where you use the induction hypothesis in your proof.

   basis: 0 is divisible by 3 ($n = 0$)

   induction: assume that $k^3 + 5k$ is divisible by 3. The goal is to show that $(k + 1)^3 + 5(k + 1)$ is divisible by 3.

   $(k+1)^3+5(k+1) = k^3+3k^2+3k+1+5k+5 = (k^3+5k)+(3k^2+3k+6)$.

   $k^3 + 5k$ is divisible by three by ind hyp, and $(3k^2 + 3k + 6)$ is evidently divisible by three, so their sum is divisible by three, which is what we needed to show.

8. Do one of the two parts. If you do both, you will get credit for your best work.

   (a) Show using the Pigeonhole Principle that if you choose 10 points from a 3 by 3 square, two of the points must be at distance $\leq \sqrt{2}$ from one another.

   Your answer should clearly indicate that you understand what the Pigeonhole Principle is.

   In the three by three square there are nine one by one squares. Placing each of 10 points in none squares, we must place at least two points in the same square. The maximum distance between two points in the same 1 by 1 square is $\sqrt{2}$, the length of the diagonal.

(b) Exhibit a bijection between the natural numbers and the integers, illustrating that these two infinite sets are “the same size”. You do not need to give a formula, just a list of the first few values making the pattern clear.

match 0 with 0, 1 with 1, 2 with -1, 3 with 2, 4 with -2, 5 with 3, 6 with -3 and so forth.