Math 187 Test 3 Review, Spring 2010

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This is a selection of relevant problems from past tests.

1. (a) How many anagrams can be made from the word ABRACADABRA? Set this up in terms of factorials then compute it.

   (b) A chain of beads is to be made with 12 beads, 4 beads of each of 3 different colors. How many different ways are there to do this? Set this up and compute it.

2. A committee with 20 members is to choose a subcommittee with 5 members. How many ways are there to do this? (Set up and compute actual number).

3. A committee with 20 members is to choose a committee with 5 members which will have a chair and a secretary. How many ways are there to do this? (Set up and compute actual numbers).

4. A committee of 12 members is to divide itself into 3 working groups each with 4 members. How many ways are there to do this if the groups are labelled Rules, Membership, Finance? How many ways are there to do this if the groups have no special identifiers? Set up and compute actual numbers.

5. As is so often the case in my tests and examples, members of a group of 21 students are all enrolled in at least one of Math, French and English. 10 students take Math. 9 students take French. 13 students take English. 3 students take Math and French. 6 students take English and Math. 4 students take English and French.

   Either compute the number of students taking all three courses or prove that I am lying. Your solution must use the inclusion-exclusion principle: set up your work in such a way that I can clearly see that you used it.

   Write the inclusion-exclusion formula for $|A \cup B \cup C \cup D|$, where A, B, C, D are any four sets.

6. How many ways are there to order four scoops of ice cream using

Baskin-Robbins's famed 31 flavors? The order of the scoops does not matter (the answer is not 31^4) and you are permitted to order more than one scoop of the same flavor.

7. Do two of the three following proofs (two are induction, one is proof by contradiction). In the math induction proofs, clearly identify the basis step, induction step, and induction hypothesis, and show me where the induction hypothesis is used in your proof.

   If you do all parts your best two will count and extra credit is possible.

   (a) Prove by mathematical induction that the sum of the first n odd numbers is equal to n^2. This sum can be written in the form $\sum_{i=1}^{n}(2i-1)$ or the form $1+3+5+\ldots+(2n-1)$

   (b) Prove by mathematical induction that for any natural number n, n^3+5n is divisible by 3.

   (c) Prove by contradiction that for any sets A and B, $(A-B)\cap(B-A)=\emptyset$. Hint: start by supposing that $(A-B)\cap(B-A)$ is not empty. Then it has an element...

8. Counting problems. This problem has four parts. Each one involves making a choice of k objects from n alternatives. Classify each problem depending on whether one can make the same choice repeatedly and whether the order in which the choices are made matters. Answer each question, showing setup and a final numerical answer.

   (a) A businessman has 10 suits. He needs to pack 3 of them to go on a trip. How many ways can he do this?

   (b) A multiple choice test has 8 questions, each with 3 possible answers. In how many ways can the test be filled out?

(c) I want to order a dozen bagels: there are poppy seed, onion and sesame bagels available today. In how many different ways can I fill my order?

(d) A bag contains 26 Scrabble tiles, each with a different letter on it. I draw 3 tiles and arrange them on the tray in front of me. How many "words" can I form in this way?

9. Give the initial setup for a proof by contradiction of the statement "If x and y are prime numbers and $x + y$ is prime, then either $x = 2$ or $y = 2$". You do not need to complete the proof, just write the beginning of the proof.

10. Prove by mathematical induction that the sum of the first n odd numbers is n^2: $1 + 3 + 5 + \ldots + (2n - 3) + (2n - 1) = n^2$. As part of your setup, write this statement in summation (Σ) notation: you do not have to use this notation in your proof, though.

    Be sure that the basis step, induction step, inductive hypothesis and any use of the inductive hypothesis in the proof are clearly identified. Make sure that you clearly distinguish between what you are assuming and what you are trying to prove.

11. Prove by mathematical induction that $n^3 + 5n$ is divisible by 3 for each natural number n.

    Be sure that the basis step, induction step, inductive hypothesis and any use of the inductive hypothesis in the proof are clearly identified. Make sure that you clearly distinguish between what you are assuming and what you are trying to prove.

12. Compute the first six terms of the sequence defined by the equations

$$a_0 = 1; a_1 = 3; a_{n+2} = 6a_{n+1} - 8a_n.$$

Derive a formula for a_n of the form $a_n = c_1 A^n + c_2 B^n$. Show complete work for the derivation. Use the formula to compute a_9.

13. You have a bin containing 12 beads of each of 4 different colors. You are going to make an open chain of just 12 beads (not 48). In each part set up your answer in terms of powers, factorials or binomial coefficients (as appropriate) then compute an actual number.

    (a) How many different chains of 12 beads with 3 of each color can you make?

    (b) How many handfuls of 12 beads of assorted colors are possible (here all that matters is how many beads you have of each color)?

    (c) How many chains of 12 beads of any mixture of colors are possible?

14. A committee of 10 people is appointing a subcommittee of 5 people. In each part, set up your calculation using binomial coefficients then compute an actual number. The last part is of course unrelated, but you still need to show your answer first using binomial coefficients then using actual numbers.

(a) How many subcommittees of 5 are possible?

(b) Suppose we also need to specify a chair and a secretary from among the members of the subcommittee. How many possible ways are there to choose a 5 member subcommittee with chair and secretary?

(c) Compute the first four terms of the expansion of $(x + y)^{15}$.

15. Of a group of 26 students, all must take English, Math, or French. 12 take English, 14 take Math, 14 take French. 7 take English and French, 3 take English and Math, 6 take French and Math. How many are taking all three subjects? You must set up and solve the problem in a way which demonstrates that you understand the Inclusion-Exclusion Principle.

16. Do one of the proofs by mathematical induction. Clearly identify the basis step, the induction hypothesis and the goal of the induction step. In the proof of the induction step, clearly indicate where the induction hypothesis is used. If you do well on both proofs, some extra credit may be possible.

   (a) Prove that the sum $1 + 3 + 5 + \ldots + (2n - 1)$ (the sum of the first n odd numbers) is equal to n^2 for each positive integer n.

   (b) Prove that $n^3 + 5n$ is divisible by 6 for each natural number n. You may use familiar properties of divisibility without proof.

17. A sequence is defined by

$$a_0 = 3; a_1 = 1; a_{n+2} = 2a_{n+1} + 3a_n$$

Compute the first six terms of this series using the rule.

Derive a formula for the nth term of this series.

Extra credit: state a recursive definition (of the same form as the definition above in this problem) for the sequence $b_n = 3^n - 2^n$.

18. You have a bin containing 13 beads of each of 4 different colors (red, green, yellow, blue). You are going to make an open chain of just 13 beads (not 52). In each part set up your answer in terms of powers, factorials or binomial coefficients (as appropriate) then compute an actual number.

    (a) How many different open chains of 13 beads with 4 red beads and 3 of each other color can you make? How many necklaces (closed chains) can you make?

(b) How many handfuls of 13 beads of assorted colors are possible (here all that matters is how many beads you have of each color)?

(c) How many open chains of 13 beads of any mixture of colors are possible?

19. A committee of 15 people is appointing a subcommittee of 5 people. In each part, set up your calculation using binomial coefficients then compute an actual number.

    (a) How many subcommittees of 5 are possible?

    (b) Suppose we also need to specify a chair and a secretary from among the members of the subcommittee. How many possible ways are there to choose a 5 member subcommittee with chair and secretary?

    (c) In how many ways can the committee of 15 divide itself into three working groups of 5? The groups do not have distinguishing names.

20. Give a proof by contradiction that no natural number n is both even and odd. You may use the fact that $\frac{1}{2}$ is not an integer. You do need to use the definitions of “even” and “odd” (this is a hint as to how to proceed). It is also important to use the fact that the integers are

closed under addition and additive inverse (or, if you prefer, that the integers are closed under subtraction). This proof does not require the well-ordering principle (though it is possible to prove this using the well-ordering principle and such a proof would be acceptable).

21. How many different anagrams can be made from the word ANAGRAMS? From the word DIFFERENT?

22. You want to make a necklace with 20 beads of 5 different colors.

    The necklace has a clasp past which beads can't be moved.

    How many ways can the necklace be made if you have 4 beads of each color?

    How does this change if there is no clasp, so beads can be moved freely all the way around the necklace?

    Suppose for the rest of the problem that you have 20 beads of each color (so you can use as many beads of each color as you like, but still no more than 20 beads in the necklace in all)?

    In how many ways can you choose a handful of 20 beads to put on the necklace (this is just a handful, order doesn't matter).

In how many ways can you make a necklace of 20 beads?

23. How many subcommittees with five members can be formed from a committee with ten members?

How many ways can the subcommittees be chosen if each subcommittee has a chair and a treasurer chosen from among its five members?

Write the first four terms of the expansion of $(x + y)^10$ (here I do want the coefficients fully computed, not just set up, and certainly not in binomial coefficient notation).

24. At a foreign language institute, 16 students study French, 19 study German, 17 study Russian, 8 study French and German, 7 study German and Russian, 9 study French and Russian. Of the 31 students, how many study all three languages?

You are required to set this up and solve it using the inclusion-exclusion method, in a way which makes it clear that you understand this method.

25. Prove by mathematical induction that the sum of the first n odd numbers is n^2.

State the theorem using summation notation, then write your proof, in which the basis step, induction step and the place where you use the induction hypothesis should all be clearly identified.

26. Prove by mathematical induction that $n^3 + 2n$ is divisible by 3 for each natural number n. The basis step, induction step and the place where you use the induction hypothesis should all be clearly identified.

27. A sequence a_n is defined by the recursive definition

$$a_1 = 3; a_2 = 5; a_{k+2} = 2a_{k+1} - a_k$$

Compute the first ten terms of this sequence.

You should recognize this sequence! Give a closed form formula for a_n. For extra credit, or to replace one of the two previous problems, prove this to be correct by math induction.

28. Freshmen at a college must take at least one of Math, English, and French in the first semester. 55 students took Math. 40 students took English. 49 students took French. 15 took both Math and English. 19 took both English and French. 17 took both Math and French. 7 students took all three courses. How many students are there in the freshman class? Use the principle of inclusion/exclusion and show your work.