Formal Arguments and Proof Strategies – Discussion and Exercises

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This document is mostly intended for reference when doing proofs. There are some exercises at the end, which are due Tuesday. I will do examples of what is expected on Monday.

Contents

1 Formal Arguments

In this document I'm going to distinguish between two kinds of letters. Letters like $P, Q, R \ldots$ will be completely unstructured sentences which are just true or false. Calligraphic letters like $\mathcal{A}, \mathcal{B}, \mathcal{C} \ldots$ are unknown expressions which might be complicated things like $P \to Q$, $(P \wedge Q) \to R$, etc. You will never have to draw this distinction on a test.

An expression like $\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3 \ldots \vdash C$ is called an *argument*. The sentences $\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3 \ldots$ we will call assumptions and the sentence C we will call the conclusion. What Marcel displays on the screen is an argument, except the assumptions appear *above* the $\vdash$ instead of the left, and the conclusion appears *below* the $\vdash$ instead of on the right.

We say that an argument $\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3 \ldots \vdash C$ is *valid* iff every assignment of values to variables in the argument which makes all the assumptions true also makes the conclusion true. Here is an example of a valid argument:

$$P, P \to Q \vdash Q.$$

An assignment which makes $P, P \to Q$ both true will make P true (obviously) and cannot make Q false, because if it did it would make $P \to Q$ false. The validity of this argument expresses the rule of *modus ponens*. An obvious example of an invalid argument is $P \vdash Q$: we can make P true (and so all the assumptions true) and still make Q false. Another invalid argument is

$$P, Q \to P \vdash Q.$$

You can check for yourself that if we make P true and Q false, then both of the assumptions are true but the conclusion is false.

An equivalent way of saying that $\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3 \ldots \vdash C$ is valid is to say that $(\mathcal{A}_1 \wedge \mathcal{A}_2 \wedge \mathcal{A}_3 \ldots) \to C$ is a tautology. So we can check validity of arguments in propositional logic (where the only variables are propositional variables $P, Q, R \ldots$ using truth tables.

2 Manipulating Arguments

2.1 Implication (IF...THEN...) in the conclusion

$\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3 \ldots \vdash \mathcal{C} \to \mathcal{D}$ is valid iff $\mathcal{C}, \mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3 \ldots \vdash \mathcal{D}$ is valid. This expresses our basic rule for proving a conclusion which is an implication, which we have already presented in board work.

In English, the strategy is “To show that a conclusion $\mathcal{C} \rightarrow \mathcal{D}$ follows (possibly from some other assumptions), add $\mathcal{C}$ to your assumptions and show that $\mathcal{D}$ follows from the new set of assumptions.

I represent this in English proofs as follows:

```
Goal:  C -> D

   Assume:  C

   Goal:  D

   <proof steps>

   D

<proving D using the new assumption C proves that C->D is true without
assuming C>
```

Notice that this is exactly what Marcel does to an argument of this form.

2.2 Conjunction (AND) in the conclusion

$\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3 \ldots \vdash \mathcal{C} \wedge \mathcal{D}$ is valid iff $\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3 \ldots \vdash \mathcal{C}$ is valid and $\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3 \ldots \vdash \mathcal{D}$ is valid.

In English, this strategy is so obvious as to be nearly invisible. "To prove that $\mathcal{C} \wedge \mathcal{D}$ follows from some assumptions, show that $\mathcal{C}$ follows from the assumptions, then show that $\mathcal{D}$ follows from the assumptions. Proving an AND statement breaks into two proofs in an obvious way: prove one part then the other.

```
Goal:  C \& D

   Goal 1:  C

   <proof steps>

   C

   Goal 2:  D

   <proof steps>

   D
```

You may have noticed in your computer proofs that this is exactly what Marcel does when you apply the `r();` command to an AND statement in the conclusion.

2.3 Disjunction (OR) in the conclusion

$\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3 \ldots \vdash \mathcal{C} \vee \mathcal{D}$ is valid iff $\neg\mathcal{D}, \mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3 \ldots \vdash \mathcal{C}$ is valid.

If $\mathcal{D}$ is a negative sentence, apply double negation: Marcel does this automatically.

In English, "to prove that $\mathcal{C} \vee \mathcal{D}$ follows (possibly from some assumptions) add $\neg\mathcal{D}$ to your assumptions and prove that $\mathcal{C}$ follows".

```
Goal:  C v D

    Assume:  ~D

    Goal:  C
```

It is an equally good strategy to do this:

```
Goal:  C v D

   Assume:  ~C

   Goal:  D
```

Marcel always does it the first way, but a `l();` command would switch the argument to the second form.

2.4 Biconditional (IFF) in the conclusion:

$\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3 \ldots \vdash \mathcal{C} \leftrightarrow \mathcal{D}$ is valid iff $\mathcal{C}, \mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3 \ldots \vdash \mathcal{D}$ is valid and $\mathcal{D}, \mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3 \ldots \vdash \mathcal{C}$ is valid. You have seen this proof strategy already in board proofs: a proof of a biconditional conclusion breaks into two parts, proofs of the conditionals in the two directions. "In order to prove $\mathcal{C} \leftrightarrow \mathcal{D}$ under some assumptions, first show that if you add $\mathcal{C}$ to your assumptions you can prove $\mathcal{D}$, then show that if you add $\mathcal{D}$ to your assumptions you can prove $\mathcal{C}$.

```
Goal:  C <-> D

   Part 1:  Assume: C

            Goal: D

   Part 2:  Assume: D

            Goal:  C
```

2.5 Negation in the conclusion

Extension of the notation: $\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3 \ldots \vdash$ means that from the assummptions $\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3 \ldots$ we can deduce a contradiction. The rules for assumptions which follow will work in the same way if the conclusion is empty as here. I'm considering a modificaton of the logical strategy (which I will describe below and which I am planning to implement in the `OneConcslusion` mode of Marcel) which will prevent this kind of argument from appearing.

$\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3 \ldots \vdash \neg\mathcal{C}$ is valid iff $C, \mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3 \ldots \vdash$ is valid [or, possibly more practically since it avoids ever introducing empty conclusions, iff $C, \mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3 \ldots \vdash \neg C$ is valid. In English, "to prove $\neg C$ from some assumptions, deduce a contradiction from C and the earlier assumptions", or "to prove $\neg C$ from some assumptions, deduce $\neg C$ from C and the earlier assumptions" (the alternative version of course shows the contradiction $C \wedge \neg C$).

```
Goal:  ~C

   Assume: C

   Goal:  contradiction <in an English argument, getting
   any contradiction A & ~A works>

   <proof steps>

   A & ~A

   <since an absurdity follows from assuming C, we have shown that
   C is false>
```

2.6 Negation in an assumption

$\neg\mathcal{D}, \mathcal{A}_2, \mathcal{A}_3 \ldots \vdash \mathcal{C}$ is valid iff $\neg\mathcal{C}, \mathcal{A}_2, \mathcal{A}_3 \ldots \vdash D$ is valid (this is an application of the contrapositive). If $\mathcal{C}$ is negative, apply double negation to $\neg\mathcal{C}$.

$\neg\mathcal{D}, \mathcal{A}_2, \mathcal{A}_3 \ldots \vdash$ is valid iff $\mathcal{A}_2, \mathcal{A}_3 \ldots \vdash \mathcal{D}$ is valid (same idea, but there is no conclusion to negate and move to the assumptions list).

In English, to use an assumption $\neg\mathcal{D}$ (and other assumptions) to prove a conclusion C, assume $\neg C$ and the other assumptions and deduce D.

```
(1)  ~D

<proof steps>

Goal: C

    Assume: ~C (for the sake of a contradiction}

    Goal: D

    <proof steps>

    D

    <once D is shown, we have the contradiction D & ~D, so we have
    shown that ~C is false, so C is true>
```

2.7 Conjunction(AND) in an assumption

$\mathcal{D} \wedge \mathcal{E}, \mathcal{A}_2, \mathcal{A}_3 \ldots \vdash \mathcal{C}$ is valid iff $\mathcal{D}, \mathcal{E}, \mathcal{A}_2, \mathcal{A}_3 \ldots \vdash \mathcal{C}$. If we assume $\mathcal{D} \wedge \mathcal{E}$, that is the same thing as assuming $\mathcal{D}$ and $\mathcal{E}$ separately.

In a proof in mathematical English:

```
<if we have this assumption>

(1) D \& E

<we can freely add these assumptions>

(2) D

(3) E
```

2.8 Disjunction (OR) in an assumption (proof by cases)

$\mathcal{D} \vee \mathcal{E}, \mathcal{A}_2, \mathcal{A}_3 \ldots \vdash \mathcal{C}$ is valid iff $\mathcal{D}, \mathcal{A}_2, \mathcal{A}_3 \ldots \vdash \mathcal{C}$ is valid and $\mathcal{E}, \mathcal{A}_2, \mathcal{A}_3 \ldots \vdash \mathcal{C}$ is valid. To use an assumption $D \vee E$ to show that C follows, assume D and show that C follows, then assume E and show that C follows.

```
(1) D v E

...

Goal: C

    By cases using the assumption (1):

    Case 1:  Assume D:

        Goal: C

        <proof steps>

        C

    Case 2:  Assume E:

        Goal: C

        <prover steps>

        C
```

2.9 Implication (IF...THEN...) in an assumption (modus ponens)

The rule usually given is "from assumptions $\mathcal{A}$ and $\mathcal{A} \to \mathcal{B}$, conclude B". This is called *modus ponens*.

Marcel does something related but a bit different, and it also reflects a more sophisticated way of reasoning with implications in mathematical English.

$\mathcal{D} \to \mathcal{E}, \mathcal{A}_2, \mathcal{A}_3 \ldots \vdash \mathcal{C}$ is valid iff $\neg\mathcal{C}, \mathcal{A}_2, \mathcal{A}_3 \ldots \vdash \mathcal{D}$ is valid and $\mathcal{E}, \mathcal{A}_2, \mathcal{A}_3 \ldots \vdash \mathcal{C}$ is valid. Think of the first part as proving $\mathcal{D}$ so that one can get the additional result $\mathcal{E}$ by *modus ponens*, and the second part as using the further result $\mathcal{E}$ to prove the final conclusion C. The assumption $\neg C$ in the first part is harmless, because if it is false the final conclusion C is of course true anyway.

We might use modus ponens in an English proof

```
(1)  A

(2)  A->B

<from the earlier assumptions (1) and (2) one can conclude>

(3)  B  <from (1), (2) by m.p.>
```

If we just have an implication in a proof, we are suddenly interested in proving the hypothesis:

```
(1)  A -> B

Goal:  C

   since we have the implication (1) as an assumption,

   Goal:  A

   <proof steps>

   A

   B <by modus ponens and the conclusion of the previous proof>

   <and then hopefully B will assist with the proof of C>
```

2.10 Biconditional (IFF) in an assumption

I'm not going to write this out in detail: an assumption $\mathcal{D} \leftrightarrow \mathcal{E}$ can simply be replaced with the two assumptions $\mathcal{D} \rightarrow \mathcal{E}$ and $\mathcal{E} \rightarrow \mathcal{D}$. A biconditional is used just like an implication, except that one can argue in both directions.

A more sophisticated application of an assumption $\mathcal{D} \leftrightarrow \mathcal{E}$ is that one can replace the sentence $\mathcal{D}$ with the sentence $\mathcal{E}$ or vice versa in any mathematical context.

2.11 Applications of the contrapositive and double negation

The equivalence of an implication $A \rightarrow B$ with its contrapositive $\neg B \rightarrow \neg A$ allows reformulations of the rules for implication.

To prove $A \rightarrow B$, one can assume $\neg B$ and deduce $\neg A$.

If one has assumptions $A \rightarrow B$ and $\neg B$, one can conclude $\neg A$. This variation of modus ponens is called *modus tollens*.

The rule of double negation allows us to give a proof strategy which can be attempted with *any* statement.

To prove $\mathcal{C}$, assume $\neg\mathcal{C}$ and deduce a contradiction. This is a proof of $\neg\neg\mathcal{C}$, which is of course equivalent to C. It is not true that any proof that has a contradiction as its goal is a proof by contradiction: a direct proof of a negative statement has a contradiction as its goal.

3 Exercises

1. (suggested activity, not graded) Go through your Marcel proofs (or make new ones) and identify uses of the various strategies described above. Remember that `r();` is always applying a conclusion strategy and `l();` is always applying an assumption strategy. Not all of them will appear (I do not think we have ever used a biconditional (IFF) assumption, for example).

2. Use a truth table to verify that $P \vee Q, \neg P \vdash Q$ is a valid argument. This is called the rule of *disjunctive syllogism*. What you need to do is show that in each line which makes both assumptions true, the conclusion is also true. Highlight the relevant lines.

3. Use a truth table to verify that $P \rightarrow Q, Q \rightarrow P$ is not a valid argument. Highlight the bad lines that show this is true. Explain what makes a bad line bad. How many bad lines to do you need to find in the truth table to show that an argument is invalid?

4. One of the arguments $P \wedge Q \vdash P \vee Q$ and $P \vee Q \vdash P \wedge Q$ is valid and one is invalid. Show that the valid one is valid and show that the invalid one is invalid, using the method of truth tables, with highlighting of relevant rows in truth tables in each case.