

Math 187 Test III

Dr. Holmes

This exam begins at 8:40 am and ends at 9:35 pm. You are permitted to use a plain scientific calculator with no graphing or symbolic computation capability. Cell phones must be turned off and out of sight. Do not pursue any extra credit opportunity until you have completed the required parts of the exam.

1. Do one of the proofs by mathematical induction. Clearly identify the basis step, the induction hypothesis and the goal of the induction step. In the proof of the induction step, clearly indicate where the induction hypothesis is used. If you do well on both proofs, some extra credit may be possible.
 - (a) Prove that the sum $2 + 4 + 6 + \dots + 2n$ (the sum of the first n even numbers) is equal to $n(n + 1)$ for each positive integer n .

- (b) Prove that $10^n - 1$ is divisible by 3 for each natural number n .
You may use familiar properties of divisibility without proof.

2. A sequence is defined by

$$a_0 = 2; a_1 = 2; a_{n+2} = 2a_{n+1} + 3a_n$$

Compute the first six terms of this series using the rule.

Derive a formula for the n th term of this series.

Extra credit: state a recursive definition (of the same form as the definition above in this problem) for the sequence $b_n = 4^n - 2^n$.

3. For each of the following sets, state whether it is a function or not. If it is not a function, give a specific counterexample showing this. If it is a function, state further whether it is a one-to-one function. If it is not a one-to-one function, give a specific counterexample showing this. If it is a one-to-one function, describe its inverse function. Give formulas for inverse functions if this is appropriate.

(a) $\{(1, 2), (1, 4), (2, 5)\}$

(b) $\{(1, 2), (2, 4), (3, 4)\}$

(c) $\{(x, y) \mid x \in \mathcal{R} \wedge y \in \mathcal{R} \wedge y = 2x - 1\}$

(d) $\{(x, y) \mid x \in \mathcal{R} \wedge y \in \mathcal{R} \wedge x^2 + y^2 = 4\}$

4. Define the set A as $\{1, 3, 5\}$ and B as $\{2, 4, 6, 8, 10\}$.
- (a) How many functions from A to B are there?

 - (b) How many one-to-one functions from A to B are there?

 - (c) How many functions from A to B are there that are onto B ?

 - (d) Give a function from A to B which is neither one-to-one nor onto B as a set of ordered pairs. Explain why it is not one-to-one, using facts about specific members of A and B . Explain why it is not onto B , using facts about specific members of A and B .

5. A permutation σ is given in array notation. A permutation τ is given in cycle notation.

$$\sigma = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 4 & 6 & 5 & 1 & 2 \end{bmatrix}$$

$$\tau = (12)(346)(5)$$

Complete the following tasks.

- (a) Present σ in cycle notation. Present τ in array notation.
- (b) Present each of σ and τ as a composition of transpositions and classify each as an even or odd permutation.
- (c) Present $\sigma \circ \tau$ in cycle notation. Present τ^{-1} in array notation.

6. In a two-foot by two-foot square we select 17 points. Explain using the Pigeonhole Principle why two of these points must be at a distance less than or equal to $\frac{\sqrt{2}}{2}$ feet from each other. A picture is fine to support your words but everything must be *said* as well as shown. You should clearly indicate where the Pigeonhole Principle is used in such a way that I can tell that you know what the Pigeonhole Principle is.

7. Write the opening part (the setup) of a proof of the following statement by contradiction. This should include the opening sentences of the proof and the goal of the part which is not shown.

If x and y are positive integers and x, y and $x + y$ are each prime, then either x or y must be equal to 2.