Math 187 Test III, Fall 2013

Dr. Holmes

November 19, 2013

The test will begin at 9 am and officially ends at 10:15 am; what will actually happen at 10:15 is that I will give a five minute warning. You are allowed your test paper, your non-graphing calculator, and your writing instrument. Your test grade will be posted by the number appearing on the first inside page of your paper.
1. Four counting problems are given. Solve each one (each is a simple one-step problem). Identify each one as a problem where order of choice matters or it does not matter, and as one where repeated choices are allowed or repeated choices are not allowed.

(a) A florist has four colors of carnations, red, white, pink, and purple. How many ways can you order a bouquet of a dozen roses?

\[
\binom{12+4-1}{12} = \binom{15}{12} = \binom{15}{3}
\]

repetitions allowed, order does not matter

(b) A committee of ten math professors need to choose a subcommittee with four members. In how many ways can they do this?

\[
\binom{10}{4}
\]

repetitions not allowed, order does not matter
(c) Seven red blue or green beads are to be put on an open chain. The two ends are different (one has a hook, the other a loop). How many ways can this be done? (You have at least seven beads of each kind available).

\[ \binom{7}{3} \text{ order matters}
\]

repetitions are allowed

(d) You have a bag with one Scrabble tile with each of the 26 letters in it. You draw three tiles and make a "word" in your Scrabble tray (any combination of letters will do). How many outcomes are possible?

\[ 26 \cdot 25 \cdot 24 = \binom{26}{3} \]

order matters

repetitions are not allowed
2. Write out the formula for \( |A \cup B \cup C| \) in terms of the sizes of the sets \( A, B, C \) and their intersections.

\[
|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + (A \cap B \cap C)
\]

There are sixteen students in the senior class at a small private high school. Every student is required to take English, French or Math. Ten students are taking English. Seven students are taking French. Eight students are taking Math. Four students are taking English and French. Four students are taking Math and French. Three students are taking English and French. How many are taking all three courses?

Write out the formula for \( |A \cup B \cup C| \) in terms of the sizes of the sets \( A, B, C \) and their intersections.

\[
16 = 10 + n + 8 - 4 - 4 - 3 + x
\]

\[
16 = 14 + x
\]

\[
\boxed{x = 2}
\]

So, students are taking all three courses.
3. Prove by mathematical induction that the sum of the first $n$ odd numbers is $n^2$:

$$ \sum_{i=1}^{n} (2i-1) = 1 + 3 + 5 + \ldots + (2n-1) = n^2 $$

Your work must clearly show the structure of an induction proof, including clearly labelled basis step, induction hypothesis, and induction goal. Indicate where in your proof of the induction hypothesis the induction goal is used.

**Basic**

$$ \sum_{i=1}^{1} (2i-1) = 1 = 1^2 $$

**Ind Hyp:** Assume $1 + 3 + 5 + \ldots + (2k-1) = k^2$

**Goal:** $1 + 3 + 5 + \ldots + (2k-1) + (2k+1) = (k+1)^2$

By (nd hyp), add $(2k+1) = 2(k+1)$ to both sides,

$$ 1 + 3 + 5 + \ldots + (2k-1) + (2k+1) = k^2 + (2k+1) $$

now $k^2 + 2k+1 = (k+1)^2$ which completes the proof.
4. Prove by mathematical induction that for each natural number \( n \), \( 5^n - 1 \) is divisible by four.

Your work must clearly show the structure of an induction proof, including clearly labelled basis step, induction hypothesis, and induction goal. Indicate where in your proof of the induction hypothesis the induction goal is used.

Bases (\( n = 0 \)): \( 5^0 - 1 = 0 \) is divisible by 4

Ind Hyp: Suppose \( 4 \mid 5^k - 1 \)

Goal: \( 4 \mid 5^{k+1} - 1 \)

Proof: By ind hyp, \( 4 \mid (5^k - 1) \)

So \( 4 \mid 5(5^k - 1) \)

So \( 4 \mid (5^{k+1} - 5) \)

So \( 4 \mid (5^{k+1} - 5) + 4 = 5^{k+1} - 1 \)

Which is what needs to be shown.
5. The sequence \( a_n \) is defined by

\[
a_0 = 3; \ a_1 = 7; \ a_{n+2} = 4a_{n+1} - 3a_n
\]

Compute the terms of this sequence up to \( a_5 \) using the recurrence relation.

\[
\begin{align*}
a_0 & = 3 \\
a_1 & = 7 \\
a_2 & = 4(7) - 3(3) = 19 \\
a_3 & = 4(19) - 3(7) = 55 \\
a_4 & = 4(55) - 3(19) = 163 \\
a_5 & = 4(163) - 3(55) = 487
\end{align*}
\]

Derive a formula for \( a_n \) for all \( n \). You can earn substantial extra credit by writing a proof by mathematical induction that your formula is correct.

\[
r^{n+2} = 4n^{n+1} - 3n^n
\]

Simplifies to \( r^2 = 4r - 3 \)

\[
r^2 - 4r + 3 = 0
\]

\[
(r-1)(r-3) = 0
\]

\( r = 1 \) or \( r = 3 \)

\[
a_n = A \cdot 1^n + B \cdot 3^n
\]

\[
a_0 = A + B = 3
\]

\[
a_1 = A + 3B = 7
\]

\[
2B = 7 \quad \text{(subtract)}
\]

\( B = 2 \)

\( A = 1 \)
6. Some functions are listed as sets of ordered pairs. For each one, indicate whether it is a function. If it is not a function, explain why not. If it is a function, indicate whether it is one-to-one. If it is not one-to-one, explain why not. If it is one-to-one, give its inverse function as a set of ordered pairs.

(a) \( \{(1, 3), (2, 2), (3, 1)\} \)

is a function is one to one

because \( \{(3, 1), (2, 2), (1, 3)\} \)

(b) \( \{(1, 2), (2, 1), (3, 2)\} \)

is a function is not 1-1

because \( f(1) = f(3) \)

(c) \( \{(1, 2), (1, 3), (2, 3)\} \)

is not a function because

1 would be mapped to both 2 and 3

How many functions are there from a set with 3 elements (such as \( \{a, b, c\} \)) to a set with 5 elements (such as \( \{1, 2, 3, 4, 5\} \))?  

\[ 5^3 = 125 \]

How many of them are one-to-one?

\[ 5 \cdot 4 \cdot 3 = 60 \]
7. Show a calculation determining \( \text{gcd}(16, 37) \), in addition determining integers \( x \) and \( y \) such that \( 37x + 16y = 1 \). You should clearly label the gcd, \( x \) and \( y \).

\[
\begin{array}{cccc}
37 & 1 & 0 & 1 \\
16 & 0 & 1 & \\
37 - 2(16) & 5 & 1 & -2 \\
16 - 3.5 & 1 & -3 & 7 \\
\end{array}
\]

\[
(-3) \cdot 37 + 7 \cdot (16) = 1 \\
\text{gcd}(37, 16)
\]

9
8. Fill in the multiplication table for arithmetic mod 7. In addition, make a table of reciprocals (multiplicative inverses) of the nonzero numbers in mod 7 arithmetic.
9. More modular arithmetic

(a) State the multiplicative inverse of 16 mod 37 (hint: you have already computed it in another problem).

\[
\begin{align*}
9 & \quad \text{from \#7} \\
-3 \cdot 37 + 7 \cdot 16 &= 1 \\
\text{so} \quad 7 \cdot 16 &\equiv 1 \pmod{37} \\
\text{so} \quad 16^{-1} &\equiv 7 \pmod{37}
\end{align*}
\]

(b) Solve the equation \(16x + 12 = 9\) in mod 37 arithmetic.

\[
\begin{align*}
16x + 12 &= 9 \pmod{37} \\
16x &= -3 \pmod{37} \\
16x &= 34 \pmod{37} \\
\overbrace{x = 7 \cdot 34 = 238 = 16} \pmod{37} \\
&\text{multiply both sides by 7} \quad \text{where} \ 7 = 16^{-1}
\end{align*}
\]