A Proof About Symmetric Difference

Dr. Holmes

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Here is the proof I did in class on which I promised you notes. I did the proof on the board “cold”, not from notes, so the proof I write here may not be identical in all details [I think after writing it that it is basically the same as my board notes].

I’m writing down all definitions that I use; you would not need to write all of these in your own proof.

I’m going to write a Marcel proof of this (Marcel can handle all the reasoning involved, including the definitions and the facts about sets and it might be interesting to see that). I’ll post this when it is ready.

This proof is extremely detailed, not because this result is terribly hard (it is rather obvious), but because the detailed proof gives examples of several different proof strategy tools.

**Preliminary Definitions:** $A \cup B$ is defined as $\{x : x \in A \lor x \in B\}$. $A \cap B$ is defined as $\{x : x \in A \land x \in B\}$. $A - B$ is defined as $\{x : x \in A \land x \notin B\}$.

**Definition (symmetric difference):** The symmetric difference $A \Delta B$ of sets $A$ and $B$ is defined as $(A - B) \cup (B - A)$.

**Theorem:** For any sets $A$ and $B$, $A \Delta B = (A \cup B) - (A \cap B)$.

**Proof Strategy Note:** Remember that to prove a theorem $A = B$ where $A$ and $B$ are sets, we first let $x$ be an arbitrarily chosen element of $A$ and show that $x \in B$ follows, then let $x$ be an arbitrarily chosen element of $B$ and show that $x \in A$ follows.

**Part I**

Let $x$ be chosen arbitrarily
Assume that (1) $x \in A\Delta B$

Goal (of Part I): $x \in (A \cup B) - (A \cap B)$

Rewrite Goal: What we need to show is that $x \in A \cup B$ and $x \notin A \cap B$.

By (1) and the definition of symmetric difference we have $x \in (A - B) \cup (B - A)$, whence by the definition of union we have $x \in A - B$ OR $x \in B - A$.

Proof Strategy Note: The main strategy for using an assumption $A \lor B$ in proving a goal $G$ is proof by cases: first prove the goal $G$ using the assumption $A$, then prove the goal $G$ again using the assumption $B$; you have to do this because you don’t know which of the two parts of the OR statement holds, so you have to show that the goal follows in either case.

proof continues: We have two cases: in each case we need to show the rewritten goal:
Case 1: (2a) $x \in A - B$

By (2a) and the definition of set difference we have (3a) $x \in A$ and (4a) $x \not\in B$. We have two things to show (from the rewritten goal) – that $x \in A \cup B$ and that $x \not\in A \cap B$. Because $x \in A$, it is true that $x \in A$ or $x \in B$, so it is true that $x \in A \cup B$.

Now we need to show that $x \not\in A \cap B$. To show this, assume $x \in A \cap B$ and reason to a contradiction: $x \in A \cap B$ means $x \in A$ and $x \in B$, but notice that (4a) $x \not\in B$ holds, so we have a contradiction. This completes the proof of the goal in Case 1.

Case 2: (2b) $x \in B - A$

The proof of Case 2 goes basically exactly as the proof of Case 1 does with the roles of $A$ and $B$ interchanged. Try writing it out yourself.

The proof of Part I is complete.

Part II

Let $x$ be chosen arbitrarily.

Assume (1) $x \in (A \cup B) - (A \cap B)$.

Goal: $x \in A \Delta B$

Rewrite Goal: by definition, the goal is equivalent to $x \in A - B$ OR $x \in B - A$.

Proof Strategy Note: To prove a goal of the form $A \lor B$, use the equivalence of $A \lor B$ with $\neg A \rightarrow B$ (which you can verify using a truth table): the way to prove that statement is to assume $\neg A$ and adopt $B$ as the new goal. Symmetrically, you can also assume $\neg B$ and adopt $A$ as the new goal. This is an alternative: it is pointless to do both.

Assume (2) $x \not\in A - B$

Goal: $x \in B - A$

Rewrite Goal: By the definition of symmetric difference, the goal is equivalent to $x \in B$ and $x \not\in A$, so it suffice to prove these two things.

By (1) and the definition of symmetric difference we have (3) $x \in A \cup B$ and (4) $x \not\in A \cap B$.  

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The assumption \( x \not\in A - B \) is equivalent to \( \neg(x \in A \land x \not\in B) \), which is by deMorgan’s law equivalent to \( x \not\in A \lor x \in B \). Since we have an assumption which is an OR statement, we can argue by cases: in each case we need to show the two things in the rewritten goal.

**Case 1:** (2a) \( x \not\in A \)

By (1) we have (3) \( x \in A \cup B \) and (4) \( x \not\in A \cap B \).

We want to show \( x \in B \): notice that we have \( x \in A \) or \( x \in B \) (by (3) and the definition of union) and we have \( x \not\in A \), so we have \( x \in B \) by the rule of disjunctive syllogism (which says that if we have \( A \lor B \) and \( \neg A \), then we can deduce \( B \)).

We also need to show \( x \not\in A \): suppose for the sake of contradiction that we have \( x \in A \); then since we just showed \( x \in B \) we would also have \( x \in A \cap B \) and this contradicts (4).

This completes the proof of Case 1.

**Case 2:** (2b) \( x \in B \)

We want to show \( x \in B \): and we assumed it, so this is easy.

We want to show \( x \not\in A \): suppose for the sake of contradiction that \( x \in A \): then \( x \in A \cap B \), but this contradicts (4).

The proof of Case 2 and Part 2 and indeed the whole theorem is complete.