Math 175 Quiz, October 1 2014, with Solutions

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A tank in the shape of an inverted cone (point up) is 40 m across at the base and 60 feet tall. Its apex (the point at the top) is 3 m below the surface. It is full of water. How much work does it take to pump all the water to the surface?

The solutions follow. Please note that any solutions I post may have mistakes in them; I am not infallible. I will have more confidence in these when I have actually marked your papers.

Water weighs $1000 \frac{m}{m^3}$. The gravitational constant is $9.8 \frac{m}{s^2}$.

The best coordinate system is to measure $x$ downward from the point of the cone. We are interested in the cross-section of the cone at height $x$, whose area $A(x)$ is $\pi (r(x))^2$ where $r(x)$ is the radius of the cross section at height $x$. $r(0) = 0$ and $r(60) = 20$, so we can see that $r(x) = \frac{1}{3} x$.

The area $A(x)$ is thus $\pi \left(\frac{1}{3} x\right)^2 = \frac{\pi x^2}{9}$. The volume of a slice of thickness $\Delta x$ at height $x$ is about $\frac{\pi x^2}{9} \Delta x$. The mass of this slice is approximately $\frac{9800 \pi x^2}{9} \Delta x$ (volume times density). The force (weight) is approximately $\frac{9800 \pi x^2 \Delta x}{9}$ (the mass multiplied by $g$). The distance through which the slice is moved is $x + 3$ (from the point $x$ meters below the point of the cone to the tip of the cone, then the further three meters to the surface). The work on the slice is approximately $\frac{9800 \pi x^2 (x+3)}{9} \Delta x$ (force times distance).

The integral to be computed is

$$\int_0^{60} \frac{9800 \pi x^2 (x + 3)}{9} \, dx =$$

$$\frac{9800 \pi}{9} \int_0^{60} x^2 (x + 3) \, dx =$$
\[ \frac{9800 \pi}{9} \int_{0}^{60} x^3 + 3x^2 \, dx = \]

\[ \frac{9800 \pi}{9} \left[ \frac{x^4}{4} + 3 \cdot \frac{x^3}{3} \right]_{0}^{60} \, dx = \]

\[ \frac{9800 \pi}{9} \left( \frac{60^4}{4} + 3 \cdot \frac{60^3}{3} \right) \]

which comes to about 11.8225 billion joules on my calculator.