11.3 to 11.5 Homework Sheet (with short answers); 11.7 problems added.

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You have until Thursday to work on this sheet. The answers I have put in are in many cases not complete answers; additional supporting work would be needed on a quiz or test.

For section 11.3, two ways to estimate

\[
\sum_{i=1}^{\infty} a_i
\]

when \( a_i = f(i) \), \( f(x) \) is a positive decreasing function, and \( \int_{1}^{\infty} f(x)\,dx \) is convergent.

\[
\sum_{i=1}^{N} a_i
\]

(the sum of the first \( N \) terms) is an estimate with error bounded by

\[
\int_{N}^{\infty} f(x)\,dx
\]

\[
\sum_{i=1}^{N} a_i + \int_{N+1}^{\infty} f(x)\,dx
\]

is an estimate with error bounded by

\[
\int_{N}^{N+1} f(x)\,dx
\]

An exercise on this material (if you feel the need of another one, do the same work for \( \frac{1}{x^4} \) as well).
1. What value of $N$ do you need in order to estimate
\[
\sum_{i=1}^{\infty} \frac{1}{x^3}
\]
within .01 using the first method (that is, choose $N$ so that the first $N$ terms added together will come within .01 of the true value). What value of $N$ do you need to estimate the same sum within .01 using the second method? I believe you will actually be able to compute the estimate by the second method using your calculator ($N$ will be small enough to be manageable).

In the first version of this that I posted, the following answer was stated incorrectly: I had $\frac{1}{N^2}$ instead of $\frac{1}{2N^2}$.

The short answer (which needs supporting calculations) is that the $N$ you need for the first part is one such that $\frac{1}{2N^2} < .01$; $N = 8$ will work.

The $N$ you need for the second part is one such that $\frac{1}{2N^2} - \frac{1}{(2(N+1))^2} < .01$ [I remind you that you need calculations to justify this]: $N = 5$ works and you should be able to compute the estimate (don’t forget that an integral goes into it too as well as the sum of five terms); actually, you should be able to compute the first one too.

For section 11.4, do the following problems. Where no specific instructions are given, determine whether the series satisfies the conditions of the Alternating Series Test (are the terms of alternating signs? do their absolute values decrease? do their absolute values converge to zero?). Does the series converge (it might converge anyway)?

1. 
\[
\sum_{n=1}^{\infty} (-1)^{n+1} \frac{\sqrt{n}}{2n + 1}
\]

It satisfies the conditions and converges. Hint: take a derivative and check signs.

2. 
\[
\sum_{n=1}^{\infty} (-1)^{n+1} \frac{n^2}{2n + 1}
\]

This diverges because the sequence of terms does not converge to zero.
3. \[ \sum_{n=1}^{\infty} \frac{(-1)^n n + 1}{4n} \]

This diverges because the sequence of terms does not converge to zero.

4. \[ \sum_{n=2}^{\infty} \frac{(-1)^n \ln(n)}{n} \]

This satisfies the conditions and converges (take a derivative to verify).

5. Approximate the value of this sum with an error less than .001:

\[ \sum_{n=1}^{\infty} \frac{(-1)^n + 1}{n^3} \]

You need an \( N \) such that \( \frac{1}{(N+1)^3} < .001 \). Add up the first \( N \) terms to get your estimate.

6. Approximate the value of this sum with an error less than .001:

\[ \sum_{n=1}^{\infty} \frac{(-1)^n + 1}{10^n + 1} \]

You need an \( N \) such that \( \frac{8}{10^{N+1}} < .001 \): \( N = 4 \) ought to work. Add up the first four terms to get your estimate.

For section 11.5, use the Comparison Test or the Limit Comparison Test to determine whether each sequence converges or diverges (by comparison with a sequence whose status you already know such as a geometric series or \( p \)-series). You need to explicitly tell me what series you are comparing it with, what the order relation is between the two series and of course what conclusion you draw.

The Limit Comparison Test: if \( \{a_n\} \) and \( \{b_n\} \) are positive series and \( \lim_{n \to \infty} \frac{a_n}{b_n} \) is a positive real number, then \( \sum_{n=1}^{\infty} a_n \) converges if \( \sum_{i=1}^{\infty} b_i \) converges and \( \sum_{n=1}^{\infty} a_n \) diverges if \( \sum_{i=1}^{\infty} b_i \) diverges (and vice versa).

1. \[ \sum_{n=1}^{\infty} \frac{1}{n^3 + 1} \]

Converges by comparison with \( \frac{1}{n^3} \).
2. $\sum_{n=1}^{\infty} \frac{1}{n + \sqrt{n}}$
Diverges by comparison with $\frac{1}{n}$ (or $\frac{1}{2n}$).

3. $\sum_{n=1}^{\infty} \frac{\sqrt{n}}{n + 3}$
Diverges by comparison with $\frac{1}{\sqrt{n}}$. You need either limit comparison or algebraic cleverness.

4. $\sum_{n=1}^{\infty} \frac{\arctan(n)}{n^2 + 1}$
Converges by comparison with $\frac{1}{n}$. Hint: the arc tangent function is bounded.

5. $\sum_{n=1}^{\infty} \frac{n}{n^2 + 1}$
Diverges by comparison with $\frac{1}{n}$ (but you need either a trick or limit comparison).

6. $\sum_{n=3}^{\infty} \frac{3 + \cos(n)}{n^2 - 4}$
Hint: I would do this in two steps, the first one getting rid of the cosine. Converges by comparison with $\frac{1}{n}$, but some cleverness is needed. Remember that the cosine is bounded.

11.7 problems (notice you also have problems from the Schaum’s outline to do).
Determine whether each series converges or diverges using the Ratio or Root test. In some cases the Ratio or Root test will give no information; say why, and use some other method to determine whether the series converges.

1. $\sum_{n=0}^{\infty} \frac{n}{2^n}$
2. \[ \sum_{n=1}^{\infty} \frac{n!3^n}{10^n} \]

3. \[ \sum_{n=1}^{\infty} \left( \frac{n}{2n+5} \right)^n \]

4. \[ \sum_{n=1}^{\infty} \frac{n+5}{n^3} \]

5. \[ \sum_{n=1}^{\infty} \frac{\ln(n)}{e^n} \]