Worksheet with Solutions

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1. Estimate $\sqrt{15.9}$ by using a linear approximation to the function $f(x) = \sqrt{x}$ near $x = 16$, or (equivalently) by using differentials.

   Define $f(x)$ as $\sqrt{x}$. $f(16) = 4$. $f'(x) = \frac{1}{2\sqrt{x}}$. $f'(16) = \frac{1}{2\sqrt{16}} = \frac{1}{8}$. The tangent line to the graph of f at $(16, 4)$ is the graph of $y - 4 = \frac{1}{8}(x - 16)$, which is the graph of the function $L(x) = 4 + \frac{1}{8}(x - 16)$ (the linear approximation). The approximation is then $L(15.9) = 4 + \frac{1}{8}(15.9 - 16) = 3.9875$.

   Using differentials: $\Delta f = \sqrt{x + dx} - \sqrt{x}$ is approximated by $df = f'(x)dx = \frac{1}{2\sqrt{x}}dx$. With the specific values $x = 16, dx = -0.1$, we get $\sqrt{16 - 0.1} - \sqrt{16}$ approximated by $\frac{1}{2\sqrt{16}}(-0.1)$, so $\sqrt{15.9} - 4$ is approximated by $-.0125$, so $\sqrt{15.9}$ is approximated by $4 - 0.125 = 3.9875$.

2. The conditions of the Mean Value Theorem apply to one of the functions listed below on the associated interval, and do not apply to the other. Identify the function and interval to which the theorem does not apply, and explain why not. For the other function, find a value of c satisfying the conclusion of the theorem.

   (a) $|x|$; $[-1, 2]$

   This function does not satisfy the conditions of the Theorem, because $|x|$ does not have a derivative at 0, and 0 is in the interval $[-1, 2]$.

   (b) $x^{\frac{3}{2}}$; $[1, 4]$

   This function is continuous and differentiable on the interval $[1, 4]$ so the theorem applies. Setting $f(x) = x^{\frac{3}{2}}$, $a = 1$, $b = 4$, we want to solve $f'(c) = \frac{f(b)-f(a)}{b-a}$, that is $\frac{3}{2}x^{-\frac{1}{2}} = \frac{8-1}{4-1} = \frac{7}{3}$, so $x^{-\frac{1}{2}} = \frac{14}{9}$, so $x = \frac{9^2}{14^2} = \frac{81}{196}$.

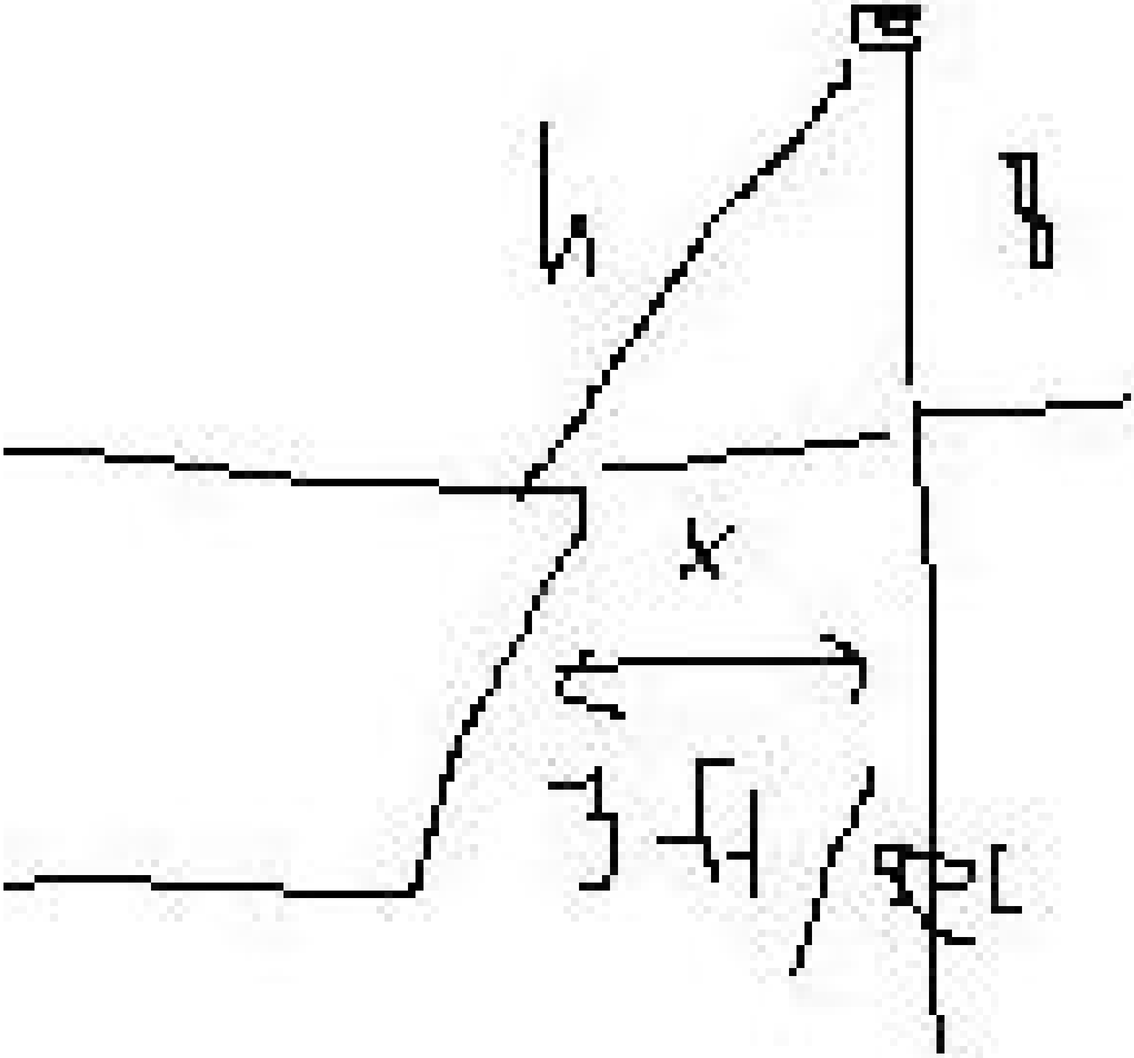
3. Related rates problems. You may choose one. If you do both parts, the best one will count.

   (a) The deck of a boat is level with the dock. A rope passes from the prow of the boat over a pulley 8 feet above the end of the dock. The boat is drifting away from the dock at 3 feet per second. The rope pays out freely from a coil of rope on the deck as the boat moves away. How fast is the rope paying out when the boat is 10 feet from the dock? You may assume (unrealistically) that the length of rope from the pulley to the prow of the boat is a straight line. There are no tides. Give your answer to as many decimal places of accuracy as your calculator provides.

   let x be the distance from the prow of the boat to the dock We know that $\frac{dx}{dt}$ is 3. Let h be the distance from the prow of the boat to the pulley (the length of the rope). We are looking for $\frac{dh}{dt}$ when $x = 10$.

   By the Pythagorean theorem, $x^2 + 8^2 = h^2$. Differentiation with respect to t gives $2x\frac{dx}{dt} = 2h\frac{dh}{dt}$. At the moment of interest, $x = 10$ and we can solve for h: $10^2 + 8^2 = h^2$ so $h = \sqrt{164}$.

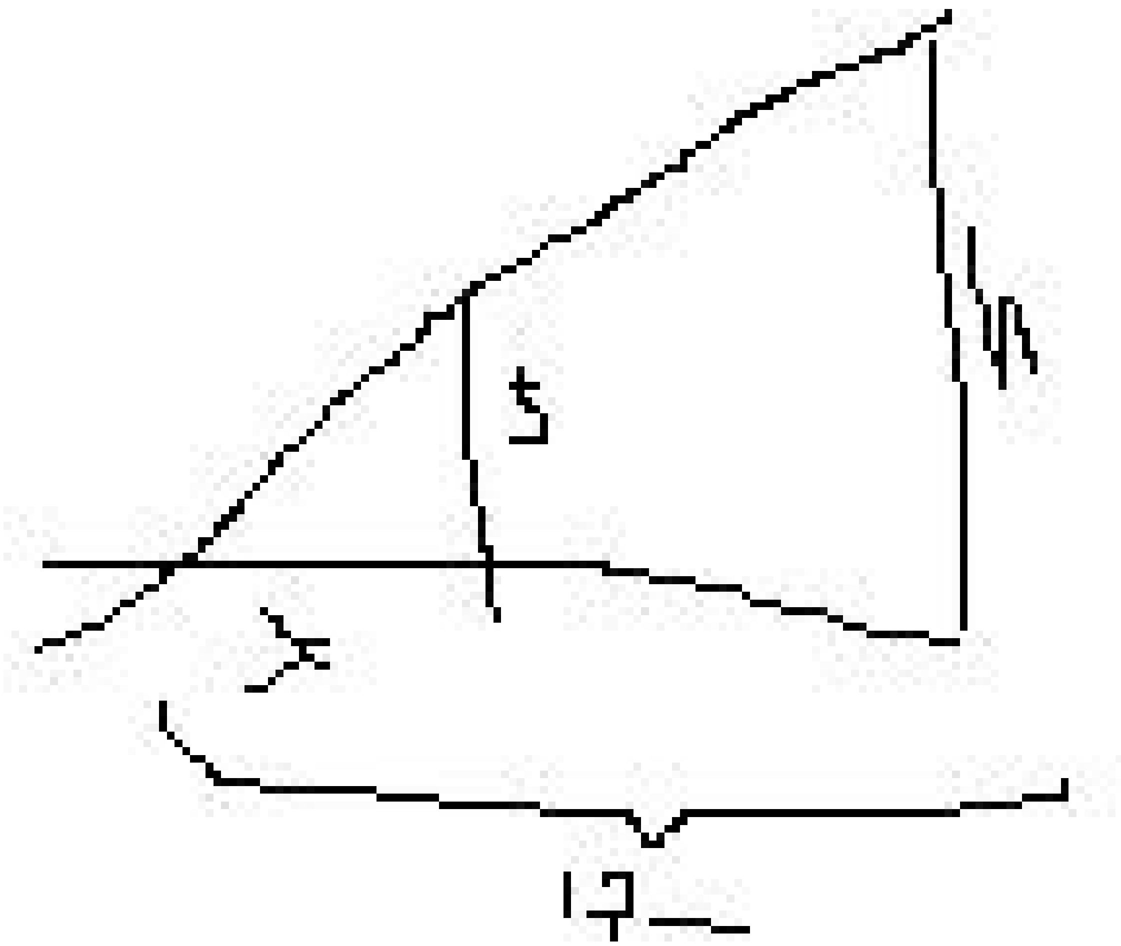
   Now we have $2(10)(3) = 2\sqrt{164}\frac{dh}{dt}$ so $\frac{dh}{dt} = \frac{30}{\sqrt{164}} = 2.34\text{ft/sec}$.



(b) A searchlight rests on the ground, 12 feet from the wall of a tall building. A five-foot tall woman walks toward the searchlight at 4 feet per second. How fast is her shadow moving up the wall when she is 6 feet from the searchlight?

Let x be the distance from the woman to the searchlight. Then $12 - x$ is the distance from the woman to the wall. We know that $\frac{dx}{dt} = -4$. Let h be the height of her shadow. We want to find $\frac{dh}{dt}$

when $x = 6$. Similar triangles (I will try to include a pic, or I will put one on my door) show that $\frac{h}{12} = \frac{5}{x}$ (the height of her shadow is to the distance 12 from the searchlight to the building as the height 5 of the woman is to her distance from the searchlight). Differentiation gives $\frac{1}{12}\frac{dh}{dt} = -\frac{5}{x^2}\frac{dx}{dt}$. Substitution of known values gives $\frac{1}{12}\frac{dh}{dt} = -\frac{5}{6^2}(-4)$. Thus $\frac{dh}{dt} = -\frac{60}{6^2}(-4) = \frac{10}{3}$ ft/sec.



4. A function and its first and second derivatives are given. Use this information to determine all intercepts, asymptotes, intervals of increase and decrease, local maxima and minima, intervals of concavity up and down, and points of inflection. Then sketch the graph of the function, labelling all signficant points with x and y coordinates. Show all work supporting your results. $f(x) = \frac{x}{(x+2)^2}$

   $f'(x) = \frac{2-x}{(x+2)^3}$

   $f''(x) = \frac{2x-8}{(x+2)^4}$

   The graph of f has a horizontal asymptote at 0: the degree of the numerator is less than the degree of the denominator.

   f itself is equal to 0 at 0 (x and y intercept) and undefined (vertical asymptote) at -2.

	$(-\infty, -2)$	$(-2, 0)$	$(0, \infty)$
x	$-$	$-$	$+$
$(x+2)^2$	$+$	$+$	$+$
f	$-$	$-$	$+$

   f' is equal to 0 at 2 and undefined at -2

	$(-\infty, -2)$	$(-2, 2)$	$(2, \infty)$
$2-x$	$+$	$+$	$-$
$(x+2)^3$	$-$	$+$	$+$
f'	$-$	$+$	$-$
f	*dec*	*inc*	*dec*

   local max at $(2, \frac{2}{(2+2)^2}) = (2, \frac{1}{8})$

   f' is equal to 0 at 4 and undefined at -2

	$(-\infty, -2)$	$(-2, 4)$	$(4, \infty)$
$2x-8$	$-$	$-$	$+$
$(x+2)^4$	$+$	$+$	$+$
f''	$-$	$-$	$+$
f	*concdown*	*concdown*	*concup*

   point of inflection at $(4, \frac{4}{(4+2)^2}) = (4, \frac{1}{9})$.