The Consistency of Quine's NF

The apparatus of type-free logic (λ-calculus or A_0) is used to prove the consistency of Quine's [2-7] New Foundations.

(I) We shall write out NF in A_0 thus. A_0 variables will be NF terms. If b and d are terms then bd (read: "d is an element of b") to be an (elementary) (NF) formula. If A and B are formulas then A → B, A & B, A v B, ~A, (Vx)A etc are NF-formulae. We introduce the concept of a stratifiable formula in the usual way [2-7]. As logical axioms and deduction rules we take the postulates of the system H of predicate logic [8]. The following are the non-logical axiom schemes of NF

13 $A(f,g) \approx (\Pi\lambda z \equiv (fz)(gz)) \rightarrow (\Pi\lambda t \equiv (tf)(tg))$; $z,t, \sim_\epsilon f,g$

14 $\lambda y\Pi\lambda x \equiv (yx)G$ (the axiom scheme of comprehension: $\sim y\epsilon G$ and $y\neq x$).

In all the postulates, A B C D are formulae, G a stratified formula, b,g and f are terms, and $E \equiv F$ iff $E \rightarrow F$ & $F \rightarrow E$. LNF shall denote a calculus of the Jensen type corresponding to NF (The sequent →E can be derived in LNF iff E is a theorem of NF) We consider each derivation C in NF with minimal use of scheme 13 (Thus, in A(f,g) terms f and g are distinct; if → A(f,g) occurs in C then → A(g,f) is not an axiom in C but is derived from the former). wlog A → A is an axiom only if A is an elementary formula.

Interpretation of LNF in the λ-calculus A_0

In the class of all deductions of A_0 we now define the subclass ANF in which LNF is interpreted. Let C be a deduction in LNF of the sequent $H \,\underline{.}\, \Delta \rightarrow \Delta$ which ends with the sequent $E \,\underline{.}\, \Gamma \rightarrow \Theta$. In exactly the same way as N-formulae were introduced in [1] we define the C formulae by creating them from N' formulae of the form Qxx. Class U derivations we shall call C-complete iff for any $\underline{C}$-formula of B, one of $\rightarrow B$ or $B \rightarrow$ is deducible in $\underline{U}$. By induction on the rank of B all deductions of A_0 are C-complete (the deductions being constructed like L-deductions)

For every formula A of NF we form a C formula $A^{\sim}$ by replacing every atomic formula $u_1 u_2$ by $u^{\sim}_1 u^{\sim}_2$ where $u^{\sim}_i$ is $u^1 \leftrightarrow \lambda v{\sim}(Q(uu))$ or $u^2 \leftrightarrow \lambda v(Q(uu))$ (subscripts omitted) and v is a new variable. By virtue of the C-completeness of stratified formulae G (see scheme 14) either $G^{\sim}\rightarrow$ or $\rightarrow G^{\sim}$ is deducible in A_0. We therefore deduce the sequent $\rightarrow G^{\sim} \underline{=} x^{\sim}y^{\sim}$ where $y^{\sim}$ is $\lambda vQyy$ or $\lambda v{\sim}(Q(yy))$

If, in scheme 13 $f^{\sim} \doteq f^2$ and $g^{\sim} \doteq g^1$ or $f^{\sim} \doteq f^1$ and $g^{\sim} \doteq g^2$, then, in A_0 we deduce $(f^{\sim}z^{\sim}) \underline{=} (g^{\sim}z^{\sim}) \rightarrow$. In the other case we examine the conversion $t^{\sim}f^{\sim} \leftrightarrow t^{\sim}g^{\sim}$. In the first case (when $(f^{\sim}z^{\sim}) \underline{=} (g^{\sim}z^{\sim}) \rightarrow$ is deduced) we first rearrange the deduction of C so that, before applying the quantifier rule (for example we take $*\Pi$ with major premiss $\Pi\lambda ga$) we substitute in C for the formula containing f and from which by use of $*\Pi$ there follows $\Pi\lambda ga$ (which does not contain any occurrences of f), a formula obtained from the first substitution of g for f, the sequent H is derived from sequents of the form $\rightarrow b$ where b is a conclusion of the corresponding implication of scheme 13 or b has the form $\Pi\lambda x \underline{=} (yx)G$ (see scheme 14) with possible relettering in H

Each wff b of the proof C (see below for dfn of class ANF) we replace by a wff b^* such that either b^* is b or we construct b^* so as to retain those subsitutions in the LNF axioms and so that alphabetic variants of wffs in C have been replaced by wffs which are interconvertible, all provable sequents $R_1, R_2, \ldots, R_n$ have been converted to $R_1^*, R_2^* \ldots . R_n^*$ to which all A_0 system rules and the rule of cut which coincide with the rules of *C* are applicable. (...stuff in brackets here.......)

By induction over $\alpha \overset{\simeq}{\leftrightarrow} \Phi[A]$ we show that in A_0 there is a derivation *B* of sequent D $\underset{\cdot}{-}$ NZ_0,E* which, like each of its subconclusions is considered to belong to ANF: N here is an arithmetical operator [1,8-13]

Let α = 1. First of all we compare the proof of each sequent E* with each axiom of the NLF calculus. In the axiom scheme A→A we compare the proof S^1 of sequent A* → A* constructed according to the rule * by virtue of the conversion A* ≈ A*. If in the scheme of the axiom of extensionality $f^{\sim} \underset{\cdot}{-} f^i$ and $g^{\sim} \underset{\cdot}{-} g^i$, i = 1,2 then, in view of $t^{\sim}f^{\sim} \approx t^{\sim}g^{\sim}$ we deduce

$$\frac{\frac{\frac{\rightarrow (t^{\sim}f^{\sim}) \underline{=} (t^{\sim}g^{\sim})}{\rightarrow (\lambda t)(t^{\sim}f^{\sim} \underline{=} t^{\sim}g^{\sim})t}}{\rightarrow \Pi\lambda t(t^{\sim}f^{\sim} \underline{=} t^{\sim}g^{\sim})}}{\rightarrow(\Pi\lambda z(f^{\sim}z^{\sim} \underline{=} g^{\sim}z^{\sim}) \rightarrow (\Pi\lambda t(t^{\sim}f^{\sim} \underline{=} t^{\sim}g^{\sim})}$$

Otherwise we continue the proof of sequent $(g^{\sim}z^{\sim}) \underline{=} (f^{\sim}z^{\sim}) \rightarrow$ to $(f^{\sim}z^{\sim}) \underline{=} (g^{\sim}z^{\sim}) \rightarrow [\Pi\lambda z(f^{\sim}z^{\sim} \underline{=} g^{\sim}z^{\sim}) \rightarrow \Pi\lambda t\ (t^{\sim}f^{\sim} \underline{=} t^{\sim}g^{\sim})]$. We compare each of the proofs (in the appropriate case) with the axiom of extensionality and we call it S^{13}. With the hypothesis $\rightarrow\Pi\lambda t \underline{=} (tf)(tg)$ (see above for the *C* premiss of the form →b) we compare the subproof of the first proof, which ends $\rightarrow\Pi\lambda t \underline{=} (t^{\sim}f^{\sim})(t^{\sim}g^{\sim}))$

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Let us consider scheme 14. Below in proof S^{14} y~*x~ is often taken instead of y*x~, the rule E* is applied in respect of variable y* instead of wff y~); In that case we define the concepts in parag. 3 such that each *F*-formula and *F*-term is an

N-formula and N-variable respectively. With the hypothesis $\rightarrow \Pi\lambda x. \doteq (yx)G$ of the C proof we compare the subproof of proof S^{14} which ends $\rightarrow \pi\lambda x. \doteq (y\tilde{}x\tilde{})G\tilde{}$. As shown there is a proof of the sequent $\rightarrow (y\tilde{}x\tilde{}) \doteq G\tilde{}$ For the new variable we get

$$\frac{\dfrac{\rightarrow \Pi\lambda x G\tilde{} \doteq (y\tilde{}x\tilde{})}{\rightarrow (\lambda y^*.\Pi\lambda x.G\tilde{} \doteq (y^*x\tilde{}))y\tilde{}}}{\rightarrow E\lambda y^*\Pi\lambda x\ (y^*x\tilde{}) \doteq G\tilde{}}$$

This proof we call S^{14} and compare with the scheme of comprehension.

The proof of sequent E^* is thus compared with each LNF axiom. For construction of the required B proofs S^L, S^{13}, S^{14} we continue to sequent D according to the rule of *K. This concludes the examination of the case α=1.

Let α=k+1 k>0. If sequent E was obtained in A_0 according to a rule [upside down Δ] other than cut, we then construct proof B based on induction hypothesis α using structure rules and the [upside-down Δ] rule of the A_0-system.

Let E and U be obtained by the cut rule from sequents $E_1 \doteq \Gamma' \rightarrow \Theta', d$ and $E_2 \doteq d, \Gamma'' \rightarrow \Theta''$. According to the induction hypothesis in ANF we can find proofs of sequents $D_1 \doteq NZ_0, \Gamma'^* \rightarrow \Theta'^*, d$ and $D_2 \doteq NZ_0, d^*, \Gamma''^* \rightarrow \Theta''^*$. We take the variable y and z which do not occur in d* then d* coincides with [h/y]d* for each wff h. In turn we get the proofs of sequents $[z/y]d^* \rightarrow [oz/y]d^*$ and $\rightarrow \Pi\lambda z P[z/y]d^*[oz/y]d^*$. From the last sequent and D_1 we derive $D^3 \doteq NZ_0, \Gamma^* \rightarrow \Theta^*, B$ where $B \simeq \&[Z_0/y]d^*(\Pi\lambda z P[z/y]d^*[oz/y]d^*)$ according to the rule &*. By the rule of transposition we transform sequent D_2 to $D_2' \doteq [Z_0/y]d^*, NZ_0, \Gamma''^* \rightarrow \Xi''^*$. We substitute $c \simeq \lambda y d^*$, $b \simeq \lambda t t Z_0$ and $a \simeq \lambda y\ \&\ (yZ_0)(yZ_0)(\Pi\lambda z P(yz)(y(oz)))$. Then $bc \approx [Z_0/y]d^*$; $ac \approx B$ and $\Xi ab \approx NZ_0$. It therefore follows from D^3 and $D^1{}_2$ that $D_3 \doteq$ $NZ, \Gamma'^* \rightarrow \Theta'^*, ac$ and $D_2' \doteq bc, NZ_0, \Gamma''^* \rightarrow \Theta$ using *≡ and the structure rules, from those of sequents D_3 and D_2 we construct the proofs of $\exists ab, NZ_0, \Gamma'^*, NZ_0, \Gamma''^* \rightarrow \Theta'^*, \Theta''^*$ and $D \doteq NZ_0, \Gamma'^*, \Gamma''^* \rightarrow \Theta'^*, \Theta''^*$. That completes the determination of class ANF.

Theorem 1

Interpretation of LNF

The sequent $E \doteq \Gamma \rightarrow \Theta$ is deducible in LNF iff the sequent $D \doteq NZ_{0}, E^*$ is derivable in ANF.

Theorem 1 consists of two parts: if U is a proof in LNF of sequent E then sequent D defined according to U with ANF construction is deducible in ANF; If in ANF there is a proof of sequent $NZ_{0,1}$ H* where only the only formulae are h-terms, (see definition of ANF) it is possible to find a sequent E which is the concluding sequent of a certain deduction from LNF. The proof of theorem 1 is based on the definition of class ANF

Theorem 2

Interpretation of NF.

Sequent $NZ_0 \rightarrow f^*$ is deducible in class ANF iff formula f is deducible in NF.

The proof of 2 follows from NF and the construction of LNF and the constructions of LNF and NF calculi. The definition of class U ?? is called N-consistent if the sequent $NZ_0 \rightarrow$ is not deducible in U.

Theorem 3 NF is consistent iff the ANF class of proofs is N-consistent.

Theorem 3 is proved with the help of theorem 2.

III Extensions of NF.

In accordance with the construction of the ANF class of proofs, we define F-terms F-formulae SF-terms and rg(B) for each F-formula of B

(i) Any term of NF we consider to be an F-term. If A is an elementary formula of NF or Qyy, then A is an elementary F'-formula.

(ii) Wffs of the form λv.Qxx and λv~Qxx are F-terms.

(iii) If d and f are essentially F-terms then [d/y]f is a F-term

(iv) If d and g are an F-term and elementary F-formula respectively, then [d/y]g is an elementary F' formula.

(v) We find rgA ≃ 0 for each elementary F'-formula A.

(vi) If A and C are essentially F'-formulae,rg(A)=i, rg(C)= j, G is a wff having the form γAC,δλxA or ~A where Γ is P,& or V,δ is Π or E then G is an F'-formula.rg(~A) ≃ i+1, rg(γAC) ≃i+j+1, rg(δλxA) ≃ i+1

(vii) If A is an F'-formula and b ≃ A then wff b is an F-formula rg(B) ≃ rg(A). If a is not an F-formula then rg(a) ≃ 0

(viii) If A is an F-formula then λxA is an SF term. Let $A_1, \ldots A_n$, B be proofs from the system A, n>=0. and let proof B be constructed like a continuation of $A_1, \ldots . A_n$ (in each branch the continuation begins with the application of the A_0 rules to the next sequent $A_1, \ldots .. A_n$ of A_0 system rules) This we call an NF-extension if it satisfies the following conditions: (i) each of the deductive sequents is an F-formula or convertible in NZ_0; (ii) The rules *Π and E* are only used in relation to F-terms, but the rule *Ξ is used in relation to SF terms. (The major premisses of these rules are obtained from F-formulae; (iii) the rules *Q is not used in B; (iv) rg(d) > rg(e) in each derivation of the wff d of the logical rule from the wff c premiss(?). (In *Π and E*([11] p 300) we assume d is obtained from a rather than ac