Pocket Set Theory: a modest proposal

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1 Introduction

We present a set theory which might be taken to be an alternative proposal for the foundations of mathematics. It was motivated for us by a side remark of Rudy Rucker in his book *Infinity and the Mind*. It might be viewed as a proposal that the universe should be viewed as very small, but then again it might be viewed as a proposal that the continuum is extremely large. We cannot help observing that our very economical selection of axioms has the effect that some of the proofs of basic assertions are very funny.

The characteristic of this theory that the only cardinals are $\aleph_0$ and c is shared by the alternative set theory of Vôpenka, which otherwise is not similar to this at all. This theory is much more classical in flavor.

2 The development

Pocket Set Theory is a first-order theory with equality and membership as primitives.

General objects are called *classes*.

Definition: We say that a class x is a *set* just in case it is an element: x is a set iff $(\exists y.x \in y)$. We say that a class is a *proper class* iff it is not a set. It is reasonable to adopt the convention that lower case variables range over sets and upper case variables over all classes, but it is not clear that we have done this consistently in this document.

Axiom of Extensionality: Objects are equal exactly if they have the same elements: $A = B \leftrightarrow (\forall x.x \in A \leftrightarrow x \in B)$. This could be weakened to allow atoms (non-sets with no elements distinct from the empty class and from each other).

Axiom Scheme of Class Comprehension: For any formula ϕ, variable x and variable A not free in ϕ, $(\exists A.(\forall x.x \in A \leftrightarrow (\exists y.x \in y) \wedge \phi))$.

Definition: $\{x \mid \phi\}$ is the class of all sets x such that ϕ whose existence is provided by Class Comprehension.

We define certain classes without any commitment about existence.

Definition: We define $\emptyset$ as the class with no elements: $\emptyset = \{x \mid x \neq x\}$.

Definition: We define $\{x, y\}$ as $\{z \mid z = x \vee z = y\}$. This exists iff x and y are sets. We define $\{x\}$ as $\{x, x\}$. We define $\langle x, y \rangle$ as $\{\{x\}, \{x, y\}\}$: note that we do not know whether this object exists for given classes (or even sets) x and y.

Definition: A *bijection between A and B* is a class of ordered pairs each of whose elements has first projection in A and second projection in B, and in which each element of A occurs exactly once as first projection and each element of B occurs exactly once as second projection.

Definition: We say that two classes are the same size iff there is a bijection between them.

Definition: We say that a class is *infinite* iff there is a bijection between the class and one of its proper subclasses. We say that a class which is not infinite is *finite*.

Axiom of Infinite Sets: There is an infinite set, and any two infinite sets are the same size.

Axiom of Proper Classes: All proper classes are the same size, and any class the same size as a proper class is a proper class.

We claim that the axioms given so far define a set theory adequate to found almost all classical mathematics outside of set theory (though certainly different from standard set theory). This is not to say that we actually advocate such a program. The motivation is this: there are only two infinite cardinals which actually show up in mathematical experience outside of set theory (both of which are arguably found in physical reality as well): these are $\aleph_0$ and c. This theory asserts that these are the only infinite cardinals of classes: every infinite class is either a countable set or an uncountable class of the cardinality of the continuum. It takes some work to discover this.

Theorem: There is a proper class.

Proof: The Russell class $\{x \mid x \notin x\}$ does the trick.

Theorem: The empty class $\emptyset$ is a set.

Proof: Suppose that $\emptyset$ was a proper class. The Russell class would then be empty, so every set would be an element of itself. Let I be an infinite set. $\{I\}$ is a set (since it is not empty and so is not the same size as the empty class). We then have $I = \{I\}$ (since all sets have to be self-membered, every singleton must be its own sole element). But this is impossible, since a singleton is clearly not infinite. So $\emptyset$ is a set.

Theorem: For every x, the singleton set $\{x\}$ is a set.

Proof: Suppose otherwise. Choose x such that $\{x\}$ is a proper class. It follows that the Russell class is also a singleton set. Moreover, the sole member of the Russell class must be the set $\emptyset$, which is clearly non-self-membered. Now let I be an infinite set. $\{I, \emptyset\}$ is a set (because it cannot be the same size as $\{x\}$) and must be self-membered (because it is different from the supposed sole member of the Russell class). But $\{I, \emptyset\}$ cannot be equal to I because it is not infinite, and cannot be equal to $\emptyset$ because it has elements.

Theorem: For any sets x and y, $\{x, y\}$ is a set.

Proof: If $\{x, y\}$ were a proper class for some choice of x and y, then the Russell class would have exactly two elements. But the Russell class has at least three elements, namely $\emptyset$, $\{\emptyset\}$ and $\{\{\emptyset\}\}$. This establishes the theorem. Notice that essentially the same argument works for sets of any concrete finite size.

Corollary: For any sets x and y, $\langle x, y \rangle$ is a set. From this it follows that any formula $\phi(x, y)$ which is bijective in the obvious sense actually defines a bijection.

An alternative proof of pairing has been proposed by Peter Mekis. Note that we assume that we have shown that the empty set and singletons are sets in the following proof, but not that pairs are sets. This theorem and the following corollary are a digression.

Theorem: The Russell class is infinite.

Proof (after Peter Mekis): Consider the class of all nonempty sets which are not elements of themselves. We give it the briefer name R^+ (and we call the Russell class R). We show that R^+ cannot be a set. Suppose that R^+ were a set. If $R^+ \in R^+$, then by definition of R^+, R^+ is nonempty and $R^+ \notin R^+$, a contradiction. If $R^+ \notin R^+$, then either $R^+ \in R^+$ (which would be contradictory) or $R^+ = \emptyset$. But $\{\emptyset\} \in R^+$, so this is also impossible. The only possibility which remains is that R^+ is a proper class. It then follows that there is a bijection from the Russell class to R^+, and since R^+ is a proper subclass of R, this means that R is infinite.

Corollary: All proper classes are infinite.

Proof (Peter Mekis): Let X be a proper class. There must be a bijection f from R to X. The restriction of f to R^+ is a bijection from the proper class R^+ to $X - \{f(\emptyset)\}$. But this implies that $X - \{f(\emptyset)\}$ is a proper class, because it is the same size as R^+. From this it follows that there is a bijection from X to $X - \{f(\emptyset)\}$, so X is infinite.

It was surprising to us that this could be proved without first proving pairing.

Corollary: $\{x, y\}$ is a set for all sets x and y.

Proof: If some pair $\{x, y\}$ were not a set, it would be the same size as R. But a pair cannot be the same size as an infinite class, as this would enable one to define an external bijection (not necessarily even realized as a set) from the pair to a proper subclass of itself, which is obviously impossible. (The same argument shows that no standard finite collection of sets can be a proper class.)

The immediately preceding Corollary makes this much easier to prove: clearly $\{x, y\}$ is not infinite.

We return to the main development.

Theorem: A class is proper iff it is the same size as the universe (the class $V = \{x \mid x = x\}$ of all sets).

Proof: It is sufficient to show that the universal class is the same size as the Russell class. A bijection from the Russell class into the universal class obviously exists. A bijection f from the universal class into the Russell class is given by $f(x) = \{\{x\}, \emptyset\}$ (no set $f(x)$ can be self-membered by cardinality considerations, and this is clearly injective). Now define the class C as the intersection of all classes which contain every self-membered set and are closed under application of the bijection f. The map which applies f to arguments in C and fixes all other objects is a bijection from the universe to the Russell class (this reproduces the proof of the Schröder-Bernstein theorem).

We now develop the theory of ordinals.

Definition: A *relation* is a class of ordered pairs. If R is a relation, we write $x\,R\,y$ for $\langle x, y\rangle \in R$. (Note that a bijection is a relation.) A relation R is a linear order on a set X if exactly one of $x\,R\,y$, $y\,R\,x$, and $x = y$ are true for any choice of x and y in X (it is trichotomous on X), and we have $x\,R\,z$ whenever we have $x\,R\,y$ and $y\,R\,z$, for any x, y, and z in X (it is transitive on X). Further, a linear order on X is a well-ordering of X iff for any nonempty subclass Y of X there is an element $y \in Y$ such that for all z such that $z\,R\,y$, z is not in Y: such an element y is called an R-minimal element of Y.

Definition: The *membership relation* is the class of all ordered pairs $\langle x, y\rangle$ such that $x \in y$.

Definition: We say that a class x is *transitive* if for all $y \in x$ and $z \in y$, we have $z \in x$.

Definition: An *ordinal* is a class which is transitive and well-ordered by the membership relation.

Theorem: No ordinal is self-membered.

Proof: if an ordinal α were self-membered, then the class of self-membered members of α would have an $\in$-minimal element. But we cannot have a self-membered element of α which has no self-membered element of α as an element!

Theorem: Every element of an ordinal is an ordinal.

Proof: Suppose that α is an ordinal and $x \in \alpha$. We need to show that x is transitive and well-ordered by membership. Suppose $z \in y$ and $y \in x$: we get $z \in \alpha$ because α is transitive, then we get $z \in x$ because membership is a linear order on α. So x is transitive. Now observe that any element of x is also an element of α: this is enough to show that the membership relation on x is trichotomous and transitive (and so a linear order) because it has the same properties on α. Suppose that Y is a nonempty subclass of x: it is thus also a nonempty subclass of α, and so has a $\in$-minimal element (which will be in x as well as in α, of course). So x is well-ordered by membership, and so is an ordinal.

Theorem: For any ordinals α, β, exactly one of the statements $\alpha \in \beta$, $\beta \in \alpha$, and $\alpha = \beta$ will be true.

Proof: Let α and β be ordinals. If α and β are both empty, then $\alpha = \beta$ and neither of the other two statements are true. If α and β are distinct, we can suppose wlog that β is nonempty. If α is empty, then $\alpha \in \beta$ (the membership-minimal element of β is α), and neither of the other two statements are true. Now suppose that neither α nor β is empty. If they are equal, neither can be an element of the other, because no ordinal is self-membered. So we are left with the case of distinct nonempty α and β. Wlog, we can suppose that we have $\beta - \alpha$ nonempty. Let γ be an ϵ-minimal element of $\beta - \alpha$. γ has as elements only elements of β which also belong to α, and it does not belong to α. We claim that it contains all elements of β which belong to α: suppose that δ is an element of β also belonging to α which is not a member of γ: it must then contain γ as a member by trichotomy of membership on β, but this is impossible because γ does not belong to α. So γ's extension is precisely the intersection of α and β. Further, $\alpha - \beta$ must be empty, because if it were not a symmetric argument would give an element of $\alpha - \beta$ with exactly the same extension, and the two differences cannot meet. Thus we find that $\alpha = \gamma \in \beta$ in this case.

Theorem: Membership well-orders the ordinals.

Proof: We have just shown that the membership relation is trichotomous on the ordinals. It is transitive on the ordinals because ordinals are transitive classes. Let Y be a nonempty class of ordinals. If it has exactly one element, that element is membership-minimal. If it has two distinct elements α and β, suppose without loss of generality that $\alpha \in \beta$. A membership-minimal element of the nonempty intersection of Y with β exists because β is an ordinal and will also be a membership-minimal element of Y.

Theorem: The class of ordinals is an ordinal.

Proof: We have just shown that it is a transitive class which is well-ordered by the membership relation.

Corollary: The class of ordinals is not a set.

Proof: If it were a set it would be a member of itself, and no ordinal is self-membered.

Theorem: Any subclass of an ordinal α is the same size as either α or some element of α.

Proof: Define the bijection by transfinite induction on an initial segment of α in the obvious way.

Definition: Define the cardinal $|x|$ of a class x as the membership-minimal ordinal which is the same size as x.

Theorem: The universe is the same size as the class of ordinals.

Proof: Both are proper classes.

Theorem: The class of all ordinals is a cardinal.

Proof: It cannot be the same size as any previous ordinal (in the order determined by membership) because the previous ordinals are sets.

Theorem: Every class is the same size as some ordinal.

Proof: The bijection between the universe and the class of ordinals determines a bijection between the given class and some set of ordinals, which will be the same size as some ordinal by a theorem above.

Theorem: Every class can be well-ordered.

Proof: The bijection between the universe and the ordinals determines a well-ordering of the universe, which is inherited by any subclass of the universe.

Theorem: Any infinite set is the same size as the first infinite ordinal ω.

Proof: An infinite set must be the same size as some ordinal. Thus there must be an infinite ordinal which is a set. The first infinite ordinal will be a set if any infinite ordinal is a set. Since all infinite sets are the same size, the result follows.

Observation: One can prove that for any x and y, if $x \cup \{y\}$ is infinite (in our official sense: Dedekind-infinite), so is x. This means that for any finite ordinal α, $\alpha + 1 = \alpha \cup \{\alpha\}$ is also finite. This means that ω is the first limit ordinal, and is in fact the intersection of all classes which contain $0 = \emptyset$ and are closed under successor. It is easy to see that the first limit ordinal is infinite (and is the first infinite ordinal).

Theorem: The class of all ordinals is the second infinite cardinal ω_1.

Proof: The first infinite cardinal is clearly the first infinite ordinal, which we have already seen is a set. If there were another infinite cardinal short of the class of all ordinals, it would be an infinite set not the same size as ω, which is impossible.

Observation: The natural numbers have already been implemented as the finite ordinals. The positive rationals can be implemented as equivalence classes of pairs of nonzero finite ordinals understood as fractions under the usual equality of fractions. The positive reals can be implemented as nonempty proper initial segments with no maximum in the positive rationals (identified with their usual least upper bound). The general real numbers can be implemented as equivalence classes of pairs $\langle r, s\rangle$ under the equality relation expected for differences $r - s$. All of these constructions can be carried out in this theory (every individual object constructed is a countable set of previously constructed objects).

Theorem: The class of real numbers (known as "the continuum") is a proper class. The universe is the same size as the continuum.

Proof: The usual proof due to Cantor shows that the power set of the natural numbers is uncountable, and therefore a proper class. Standard arguments show that the power set of the natural numbers is the same size as the class of reals. Of course, one can use the usual proof that the reals themselves make up an uncountable set.

Theorem: Any class of real numbers is either finite, countable or the size of the continuum.

Proof: Obvious, since these are the only possible cardinals for any class whatsoever.

Now we tidy things up by proving some assertions which we are used to having as axioms which were not used in this development. In fact, we will review all the axioms of *ZFC* and indicate their status.

Extensionality: True here. Already provided as an axiom.

Pairing: Proved as a theorem above.

Union: This has not been proved or used so far, but it does hold. Briefly, every set is finite or countably infinite. A finite union of finite or countably infinite sets is certainly countably infinite. A countably infinite union of finite or countably infinite sets is countably infinite – this is proved in the usual way, but notice that Choice is needed: we need to be able to select a given order for each of the sets in an infinite union.

Power Set: This is false, since the power set of ω is not a set.

Infinity: The class of finite ordinals (the first infinite ordinal ω) is a set.

Separation: Any subclass of a finite set is finite. Any subclass of a countably infinite set is either finite or countably infinite. This is another axiom not used in this development.

Replacement: This follows immediately from the proposition that the proper classes are exactly the classes the size of the universe: this means that anything the same size as a set is a set.

Foundation: We have not asserted this, but we could. We have not needed it in our proofs; any combinatorial consequence of Foundation is true in this theory, as the class of well-founded sets is proper and so is the same size as the universe.

Our preference is to adopt an axiom of Anti-Foundation (the Free Construction Principle introduced below).

Choice: This has been proved (in the form of the assertion that there is a class well-ordering of the universe, which can be used to define well-orderings of sets or choice sets for partitions in obvious ways).

In terms of standard set theory, this is the theory of hereditarily countable sets with AC and CH. To be precise, this is Kelley-Morse set theory minus Power Set, plus CH + "the continuum is a proper class" + Global Choice.

This theory is interesting because it allows only the two infinite cardinals which demonstrably "occur in nature" (the cardinality of the natural numbers and the cardinality of the continuum). It is entirely adequate for mathematical purposes outside of set theory (some coding is required to discuss sets of reals, for example, but sets of reals which are useful in practice generally can be coded as reals, and even more easily as general HC sets). In a calculus context, note that a continuous function from the reals to the reals is a proper class, but its restriction to the rationals is a set and completely determines the function.

Note that it is crucially important that our theory of classes is impredicative (we can quantify over all classes in the definition of a class: this is used in the definition of "ordinal", for example).

The usual techniques of model theory allow us to show that any consistent theory has a set model. We adopt a further axiom which will ensure that any theory of sets (even a theory with atoms) has an inner model in this theory.

We call a relation R an *extensional relation* just in case distinct elements of the range of R have distinct preimages. Note that we allow there to be many elements of the domain of R which are not in the range (we call these "minimal" elements) not just one as a stronger extensionality property would require). A function f is said to be a partial decoration on R iff f is injective, for any a such that aRb and $f(b)$ is defined, $f(a)$ is also defined, and for any a in the range of R for which $f(a)$ is defined, $f(a) = \{f(b) \mid bRa\}$ (note that minimal elements can be decorated with nonempty sets or atoms (if there are atoms)).

Note that this is not the antifoundation axiom of Aczel. The requirement that f be injective assures that the decoration process introduces no identifications between sets decorating elements of the range of R except those that are forced by extensionality.

Axiom of Free Construction: For any extensional set relation R and partial decoration f on R, there is an extension of f which is a partial decoration on R and whose range includes both the domain and range of R.

This axiom is sufficient to ensure that any consistent set theory satisfying extensionality (even in the weak form allowing atoms) has an inner model in this theory (a model whose membership relation is actually the restriction of membership to the model). Of course such models are countable.

The axiom can probably be generalized to class relations R under which every preimage of a range element is a set.