1. (4) Find the first and second derivatives of \( w = g(z) = z^5 - 2z^3 + 7z^2 - 4 \)

\[
\frac{dw}{dz} = g'(z) = 5z^4 - 6z^2 + 14z
\]

\[
\frac{d^2w}{dz^2} = g''(z) = 20z^3 - 12z + 14
\]

2. (4) If \( y = x^3e^x \), find \( \frac{dy}{dx} \).

\[
\frac{dy}{dx} = 3x^2e^x + x^3e^x = e^x(3x^2 + x^3)
\]

3. (4) If \( h(x) = \frac{x^2 - 1}{x + 0.5} \), find \( h'(x) \).

\[
\frac{h'(x)}{(x + 0.5)^2} = \frac{(x + 0.5)(2x) - (x^2 - 1)}{(x + 0.5)^2}
\]

\[
\frac{h'(x)}{(x + 0.5)^2} = \frac{x^2 + x + 4}{(x + 0.5)^2}
\]

Please turn over.
4. (4) If \( y = \arctan \left( \frac{1}{x} \right) \), find \( \frac{dy}{dx} \).

\[
\frac{dy}{dx} = \arctan' \left( \frac{1}{x} \right) \frac{d}{dx} \left( \frac{1}{x} \right) \\
= -\frac{1}{1 + \left( \frac{1}{x} \right)^2} \left( -\frac{1}{x^2} \right) \\
= \frac{1}{x^2 + 1}
\]

5. (4) If \( k(x) = \ln(-x), x < 0 \), find \( k'(x) \).

\[
k'(x) = \frac{1}{-x} (-1) = \frac{1}{x} \quad \text{for } x < 0
\]
1. Consider the function \( y = f(x) = \frac{1}{x} + 2 \).

(a) Using the definition, calculate \( f'(x) \). Hint. Your work should include a difference quotient and a limit.

\[
 f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{\frac{1}{x+h} + 2 - (\frac{1}{x} + 2)}{h}
\]

\[
 = \lim_{h \to 0} \frac{1}{h} \frac{x - (x+h)}{x(x+h)} = \lim_{h \to 0} \frac{-h}{hx(x+h)}
\]

\[
 = -\frac{1}{x^2}.
\]

(b) Using your work from (a), calculate \( f'(-1) \).

\[
 f'(-1) = -\frac{1}{(-1)^2} = -1
\]

(c) Write an equation for the tangent line to \( y = f(x) \) at \( P(-1,1) \).

\[
 y - 1 = -1(x+1) \implies y - 1 = -x - 1
\]

\[
 y = x + 2
\]
2. Explorers on a small airless planet use a spring gun to launch a ball bearing vertically upward from the surface at a launch velocity of $15 \text{m/sec}$. If the acceleration of gravity at the planet’s surface is $g_s \text{ m/sec}^2$, the explorers calculate that the ball bearing will be at a height of $s = 15t - (g_s/2)t^2$, $t$ seconds later. The ball bearing reached its maximum height 20 seconds after being launched. What is the value of $g_s$? Hint: What happens to the velocity at the maximum height?

\[ s = 15t - \frac{1}{2} g_s t^2 \]
\[ v = \frac{ds}{dt} = 15 - g_s t \]

\[ \text{At } t = 20, \quad v = 0 \quad \Rightarrow \quad g_s = \frac{15}{20} = \frac{3}{4} \quad (\text{m/sec}^2) \]
3. If \( s \) is the distance of a heavy meteorite from the surface of the earth, \( \frac{ds}{dt} = -\frac{k}{\sqrt{s}} \), where \( k \) is a positive constant. Show that the acceleration of the meteorite is proportional to \( \frac{1}{s^2} \). Hint. The Chain Rule is involved here.

\[
\frac{ds}{dt} = -\frac{k}{\sqrt{s(t)}}
\]

\[
\text{acceleration} = \frac{d^2s}{dt^2} = \frac{d}{dt} \left( -\frac{k}{\sqrt{s(t)}} \right) = \frac{d}{dt} \left( -k \cdot s(t)^{-\frac{1}{2}} \right)
\]

\[
= -k \left( -\frac{1}{2} \right) s(t)^{-\frac{3}{2}} \frac{ds}{dt}
\]

\[
\lim_{t \to t_0} \frac{ds}{dt} = -\frac{k}{\sqrt{s}} + \varepsilon
\]

\[
\frac{d^2s}{dt^2} = \frac{k}{2} s^{-\frac{3}{2}} \left( -\frac{k}{\sqrt{s}} \right)
\]

\[
\frac{d^2s}{dt^2} = \frac{k^2}{2} \frac{1}{s^2}
\]

Thus, \( \frac{d^2s}{dt^2} \) is proportional to \( \frac{1}{s^2} \).
4. The line that is normal to the curve \( x^2 + 2xy - 3y^2 = 0 \) at \((1, 1)\) intersects the curve at what other point? Note. A normal line to a curve at a point is the line perpendicular to the tangent at that point.

\[
x^2 + 2xy - 3y^2 = 0
\]

\[
\frac{d}{dx} \left( 2x + 2y + 2x \frac{dy}{dx} - 6y \frac{dy}{dx} \right) = 0
\]

\[
\frac{dy}{dx} (2x - 6y) = -2x - 2y
\]

\[
\frac{dy}{dx} = \frac{2x + 2y}{6y - 2x}
\]

Let, slope of tangent line @ \((1, 1)\)

\[= \frac{dy}{dx} = \frac{4}{4} = 1\]

slope of normal line @ \((1, 1)\) = \(-\frac{1}{1} = -1\)

equation of normal line:

\[
y - 1 = - (x - 1)
\]

\[
y - 1 = -x + 1
\]

\[
y = -x + 2
\]

To find the intersecting point, substitute into original equation:

\[
x^2 + 2x(-x + 2) - 3(x^2 - 4x + 4) = 0
\]

\[
-4x^2 + 16x - 12 = 0 \Rightarrow x^2 - 4x + 3 = 0
\]

\[
x = 1 \text{ or } x = 3 \text{ so } x = 3, y = -1 \text{ is the point.}
5. (a) Two commercial airliners are flying at 40,000 feet along straight-line courses that intersect at right angles. Plane A is approaching the intersection point at speed of 442 miles per hour and plane B is approaching the intersection at 481 miles per hour. At what rate is the distance between the planes changing when A is 5 miles from the intersection point and B is 12 miles from it?

Let \( x, y, d \) be defined by the diagram.

\[
\begin{align*}
\frac{dz}{dt} &= x^2 + y^2 \\
2 \frac{dz}{dt} &= 2xx' + 2yy' \\
\frac{dx}{dt} &= xx' + yy' \\

d &= \frac{xx' + yy'}{\sqrt{x^2 + y^2}} \\
\frac{dx}{dt} &= \frac{5 \cdot 442 + 12 \cdot (-481)}{\sqrt{5^2 + 12^2}} = \frac{-7982}{13} \\
\frac{dx}{dt} &= -614 \text{ mph}
\end{align*}
\]

(b) Assuming that the speeds of the airliners don't change, at what rate is the distance between the planes changing when A passes through the intersection point?

At that moment, \( x = 0 \), \( y \) is some value.

\[
\frac{dx}{dt} = \frac{0 \cdot x' + yy'}{\sqrt{yx^2}} = y' = -481 \text{ mph}
\]
6. The relationship between the amplitude of an input signal and the amplitude of the output signal of a certain device is given by $A_{out} = F(A_{in})$ where we don't have an exact formula for $F$. We are, however, able to make the following measurements: $F(1) = 27$, $F(1.01) = 27.52$, and $F(0.99) = 26.49$. Give the best estimate you can for $F(1.035)$ and explain what you did.

We approximate $F'(1)$ by $\frac{F(1.01) - F(0.99)}{1.01 - 0.99}$

$F'(1) \approx \frac{27.52 - 26.49}{.02} = \frac{1.03}{.02} = 51.5$

Using differentials

$F(1.035) - F(1) \approx dF = F'(1) \cdot dh$

$\approx 51.5 \cdot .035$

$F(1.035) \approx 27 + 51.5 \cdot .035$

$= 28.8025$

I would also accept

$F(1.035) = F(1) + 3.5 \cdot (F(1.01) - F(1))$

$= 27 + 3.5 \cdot 52$

$= 28.82$

(This is essentially approximating $F'(1)$ by $\frac{F(1.01) - F(1)}{.01}$)
7. Find the absolute (global) maximum and minimum values of \( g(x) = \frac{1}{x} + \ln(x) \) on the interval \([0.5, 4]\). Sketch a graph of the function, identify the points on the graph where the absolute extrema occur, and include their coordinates.

\[
\begin{align*}
g(x) &= \frac{1}{x} + \ln(x) \\
g'(x) &= -\frac{1}{x^2} + \frac{1}{x} \\
g''(x) &= \frac{2}{x^3} - \frac{1}{x^2}
\end{align*}
\]

Endpoints: 0.5, 4

Critical points: 
\[-\frac{1}{x^2} + \frac{1}{x} = 0\]
\[-1 + x = 0 \Rightarrow x = 1\]

\[
g(0.5) = \frac{2}{0.5} + \ln(0.5) = 19.324 \\
g(1) = 1
\]

\[
g(4) = \frac{1}{4} + \ln(4) = 1.6363
\]

Global maximum value = 1.6363
(occur at \( x = 4 \))

Global minimum value = 1
(occur at \( x = 1 \))

\[
\text{GLOBAL MAX (4, 1.6363)}
\]

\[
\text{GLOBAL MIN (1, 1)}
\]

Note: \( g''(x) > 0 \) on \((0.5, 4)\)
8. Show that \( |\arctan(b) - \arctan(a)| \leq |b - a| \). Hint. What does the Mean Value Theorem say about \( \frac{\arctan(b) - \arctan(a)}{b-a} \)? Then take absolute values.

\[
\text{MVT } \Rightarrow \\
\frac{\arctan(b) - \arctan(a)}{b-a} = \frac{1}{1+c^2} \\
(\arctan'(c))
\]

\[
|\arctan(b) - \arctan(a)| = \frac{1}{1+c^2} |b-a|
\]

But \( \frac{1}{1+c^2} < 1 \) \( \Rightarrow \)

\[
|\arctan(b) - \arctan(a)| \leq |b-a|
\]
9. Find the absolute maximum value of \( h(x) = x^2 \ln \left( \frac{1}{x} \right) \) and state where it is assumed.

The domain must be \( x > 0 \)

\[
h(x) = x^2 \ln \left( \frac{1}{x} \right)
\]

\[
h'(x) = 2x \ln \left( \frac{1}{x} \right) + x^2 \cdot \left( \frac{1}{x} \right)' \left( - \frac{1}{x^2} \right) = -2x \ln(x) - 1
\]

\[
h''(x) = -2 \frac{\ln(x)}{x} - 2x \left( \frac{1}{x} \right) = -2 \ln(x) - 3
\]

critical points: \(-2x \ln(x) - 1 = 0\)

\[-2 \ln(x) = 1\]

\[\ln(x) = -\frac{1}{2}\]

\[x = e^{-\frac{1}{2}}\]

\[
\lim_{x \to 0^+} x^2 \ln \left( \frac{1}{x} \right) = \lim_{x \to 0^+} (x^2 \ln(x))
\]

\[
= - \lim_{x \to 0^+} \frac{\ln(x)}{1/x^2} = - \lim_{x \to 0^+} \frac{1}{x^2} \cdot x^{1/2} = - \lim_{x \to 0^+} e^{-1/2}
\]

\[h(e^{-\frac{1}{2}}) = e^{-1} (2) = \frac{2}{e}\]

\[
\lim_{x \to \infty} x^2 \ln \left( \frac{1}{x} \right) = -\infty
\]

\[
h''(e^{-\frac{1}{2}}) = 1 - 3 = -2
\]

So, for the absolute max of \( h(x) \) is \( h(e^{-\frac{1}{2}}) = \frac{2}{e} \) which is assumed at \( x = e^{-\frac{1}{2}} \).