Cofinalities and coinitialities
Consistency results
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The coinitiality of Efimov spaces

Cofinalities of Boolean algebras
Definition (Koppelberg)
The cofinality $\text{cf}(A)$ of a Boolean algebra $A$ is the least ordinal $\alpha$ such that $A$ is the union of an increasing chain of length $\alpha$ of proper subalgebras of $A$, provided such a chain exists.

Basic facts
If $A$ is infinite, then $\text{cf}(A)$ exists and is a regular cardinal $\leq |A|$.

If $B$ is an infinite quotient of $A$, then $\text{cf}(A) \leq \text{cf}(B)$. 
Definition
A Boolean algebra $A$ has the *countable separation property* if for every pair $(S, T)$ of countable subsets of $A$ such that every element of $S$ is disjoint from every element of $T$, there is some $a \in A$ that is above all elements of $S$ and disjoint from all elements of $T$. (Every countable gap is filled.)

Note that the countable separation property follows from $\sigma$-completeness.

Theorem (Koppelberg)

*Every infinite Boolean algebra with the countable separation property has uncountable cofinality. Every infinite complete Boolean algebra has cofinality $\aleph_1$.***
Cofinalities of $C^*$-algebras and coinitialities of compact spaces
**Definition**

The *cofinality* \( \text{cf}(A) \) of a \( C^* \)-algebra \( A \) is the least ordinal \( \alpha \) such that \( A \) has a dense subalgebra that is the union of an increasing chain of length \( \alpha \) of proper closed \( * \)-subalgebras of \( A \), provided such a chain exist.

The *coinitiality* \( \text{ci}(X) \) of a compact space \( X \) is the cofinality of the \( C^* \)-algebra \( C(X) \) of continuous functions from \( X \) to \( \mathbb{C} \).

**Basic facts**

Every infinite-dimensional \( C^* \)-algebra has a regular cofinality that is bounded by its density. Every infinite compact space has a regular coinitiality that is bounded by its weight.
Remarks

The coinitiality of an infinite compact space $X$ is the least ordinal $\alpha$ such that $X$ is the limit of a nontrivial inverse system indexed by $\alpha$.

\[ X \rightarrow \cdots \rightarrow X_{\nu+1} \rightarrow X_{\nu} \rightarrow \cdots \rightarrow X_0 \]

We will use this characterization later on.

If $X$ is infinite, compact and zero-dimensional, then $\text{ci}(X) = \text{cf}(\text{clop}(X))$, where $\text{clop}(X)$ is the Boolean algebra of clopen subsets of $X$. 
Lemma (Boolean case due to Koppelberg and van Douwen)

Let $X$ be an infinite compact space. Then the following hold:

a) If $Y$ is a closed subspace of $X$ of uncountable cofinality, then $\text{ci}(X) \leq \text{ci}(Y)$. (This is true without the uncountability assumption but harder to prove.)

b) Let $w(X)$ denote the weight of $X$, i.e., the smallest size of a basis of the topology on $X$. Then $\text{ci}(X) \leq \text{cf}(w(X))$.

c) $\text{ci}(X) \leq 2^{\aleph_0}$

d) Let $a(X)$ denote the altitude of $X$, i.e., the smallest length of a strictly decreasing sequence of closed subsets of $X$ whose intersection is a singleton. Then $\text{ci}(X) \leq a(X)$. 
Consistency results
Theorem (Koszmider)

*It is consistent that every compact zero-dimensional space has altitude at most $\aleph_1$ while $2^{\aleph_0} > \aleph_1$.*

*In particular, it is consistent that $2^{\aleph_0}$ is large while every compact zero-dimensional space has small cointiality.*

Remark

Observe that a compact space is of countable altitude iff it contains a converging sequence.
It seems to be set-theoretic folklore that under MA, every infinite compact space of weight $< 2^{\aleph_0}$ has a nontrivial converging sequence.

**Theorem**

Let $\text{cov}(\mathcal{M})$ denote the least size of a family of meager sets that covers the real line. Then every infinite compact space of weight $< \text{cov}(\mathcal{M})$ has a nontrivial converging sequence.

**Corollary**

After adding $\aleph_{\omega_1}$ Cohen reals to a model of CH, we obtain a model of set theory in which every compact space has cointinality $\leq \aleph_1 = \text{cf}(2^{\aleph_0}) < 2^{\aleph_0}$. 
Question

Is it consistent that there is a Boolean algebra of cofinality $> \aleph_1$ or a compact space of cointiality $> \aleph_1$?

What about noncommutative $C^*$-algebras?

Observe that a compact space of cointiality $> \aleph_1$ can neither contain a nontrivial converging sequence nor a copy of $\beta \omega$. In other words, it has to be an Efimov-space. However, none of the currently known constructions of an Efimov space can yield a space of cointiality $> \aleph_1$. 
Characterization of countable cofinality
Definition
Let \( X \) be a compact space. A pair \( (x^0_n)_{n \in \omega}, (x^1_n)_{n \in \omega} \) is a double sequence in \( X \) if for all \( p \in \beta \omega \setminus \omega \) the two sequences have the same \( p \)-limit.

Theorem
An infinite compact space \( X \) is of countable coinitiality iff it contains a double sequence.

This theorem implies that an infinite compact space \( X \) of coinitiality \( \geq \aleph_1 \) contains neither a double sequence nor a copy of \( \beta \omega \).
The cofinality of Efimov spaces
Theorem
Assuming ♦, there is an Efimov space of uncountable cofinality, i.e., there is an infinite compact space that contains neither a double sequence nor a copy of $\beta \omega$.

Theorem
Assuming ♦, there is an Efimov space of countable cofinality, i.e., there is an infinite compact space that contains a double sequence but no nontrivial converging sequence and no copy of $\beta \omega$. 
Thank you!