POTENTIAL CONTINUITY OF COLORINGS

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Abstract. We say that a coloring $c : [\kappa]^n \to 2$ is *continuous* if it is continuous with respect to some second countable topology on κ. A coloring c is *potentially continuous* if it is continuous in some $\aleph_1$-preserving extension of the set-theoretic universe. Given an arbitrary coloring $c : [\kappa]^n \to 2$, we define a forcing notion $\mathbb{P}_c$ that forces c to be continuous. However, this forcing might collapse cardinals. It turns out that $\mathbb{P}_c$ is c.c.c. if and only if $\mathbb{P}_c$ is potentially continuous. This gives a combinatorial characterization of potential continuity.

On the other hand, we show that adding $\aleph_1$ Cohen reals to any model of set theory introduces a coloring $c : [\aleph_1]^2 \to 2$ which is potentially continuous but not continuous. $\aleph_1$ has no uncountable c-homogeneous subset in the Cohen extension, but such a set can be introduced by forcing. The potential continuity of c can be destroyed by some c.c.c. forcing.

1. Introduction

An *n-dimensional coloring* on a set X is a map c from the set $[X]^n$ of n-element subsets of X to the set k of colors where k is a natural number > 1. 2-dimensional colorings will also be called *pair colorings*. We restrict ourselves to two colors since in many cases questions about colorings with finitely many colors can be reduced to questions about colorings with just two colors.

If X is a Hausdorff space, then the natural topology on $[X]^n$ is the topology generated by the sets

$$[O_1, \dots, O_n] = \{\{x_1, \dots, x_n\} : x_1 \in O_1 \wedge \dots \wedge x_n \in O_n\},$$

where the O_i are pairwise disjoint open subsets of X. All topological spaces are assumed to be Hausdorff.

Continuous pair colorings on Polish spaces show up in various contexts (see [5]) and admit a nice structure theory (see [4]). The starting point of the theory of continuous pair colorings on Polish spaces is Galvin's theorem (see [6, Theorem 19.7]) saying that for every uncountable Polish space X and every continuous coloring $c : [X]^2 \to 2$ there is a (non-empty) perfect c-homogeneous set $H \subseteq X$. Here a set $H \subseteq X$ is *c-homogeneous* (or just *homogeneous*) for a coloring $c : [X]^n \to 2$ if c is constant on $[H]^n$.

The natural analog of Galvin's theorem for n-dimensional colorings fails if $n \geq 3$ since there is a continuous 3-dimensional coloring on 2^ω without uncountable homogeneous sets [6, Exercise 19.10]. However, there are versions of Galvin's theorem that remain true in higher dimensions [2] (see also [6, Section 19.B]).

Continuous colorings on separable metric spaces that are not necessarily complete are probably even more interesting when it comes to applications. It turns out that for many results the metric is not needed and it is enough that the colorings under consideration are continuous with respect to a second countable topology. Such colorings have been studied for instance by Abraham, Rubin and Shelah in

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[1]. By Urysohn's metrization theorem, a second countable space is metrizable iff it is regular.

Given a pair coloring on an uncountable separable metric space X, it can happen that there is no uncountable homogeneous set. This can be seen as follows:

For $\{x,y\} \in [2^\omega]^2$ let $\Delta(x,y) = \min\{n \in \omega : x(n) \neq y(n)\}$. Define a coloring $c_{\min} : [2^\omega]^2 \to 2$ by letting $c_{\min}(x,y) = \Delta(x,y) \mod 2$ (see [4] for an explanation of the notation $c_{\min}$). It is easily checked that $c_{\min}$-homogeneous sets are nowhere dense. Hence, if $X \subseteq 2^\omega$ is an uncountable set whose intersection with every nowhere dense subset of 2^ω is at most countable, i.e., if X is a Luzin set, then there are no uncountable homogeneous sets for $c_{\min} \restriction [X]^2$. In fact, this argument goes through for every continuous pair coloring c on a Polish space without open c-homogeneous sets. Luzin sets exist under CH or after adding uncountably many Cohen reals to any model of set theory.

However, as was shown in [1], under CH the following is true: for every continuous pair coloring c on an uncountable, second countable space X there is a c.c.c. forcing that adds a countable family $\mathcal{H}$ of c-homogeneous sets that covers X. In particular, the forcing adds an uncountable c-homogeneous set.

Without CH, given a continuous pair coloring c on a second countable space X one can first collapse $2^{\aleph_0}$ to $\aleph_1$ and then add countably many c-homogeneous sets that cover X using some c.c.c. forcing. It follows that under PFA the following is true:

"For every continuous coloring $c : [X]^2 \to 2$ on a second countable space X of size $\aleph_1$ there is a countable family $\mathcal{H}$ of c-homogeneous set such that $X = \bigcup \mathcal{H}$."

Let us denote this statement by WOCA, since it is a weak form of the Open Coloring Axiom ($\mathrm{OCA}_{[\mathrm{ARS}]}$) introduced in [1].

It is clear that PFA implies the analog of WOCA for colorings on $\aleph_1$ that can be forced to be continuous with respect to some second countable topology on $\aleph_1$ by some proper forcing.

For the purpose of this article, we call an n-dimensional coloring on a set X *continuous* if it is continuous with respect to some second countable topology on X. We call an n-coloring c on a set X *potentially continuous* if there is some extension W of the set theoretic universe V such that $(\aleph_1)^V = (\aleph_1)^W$ and in W, c is continuous. The term "potentially continuous" is motivated by similar concepts that exist in model theory (see [7] and [3]).

We will show that a coloring is potentially continuous iff it can be forced to be continuous by some c.c.c. forcing. In particular, WOCA for potentially continuous colorings instead of continuous colorings follows from WOCA for continuous colorings together with $\mathrm{MA}_{\aleph_1}$, Martin's Axiom for $\aleph_1$ dense sets.

On the other hand, we give a consistent example of a pair coloring on $\aleph_1$ that is potentially continuous but not continuous and that can be forced to be not potentially continuous.

2. Forcing the continuity of colorings

Let κ be an infinite cardinal, n a natural number > 1 and $c : [\kappa]^n \to 2$. We define a forcing notion $\mathbb{P}_c$ of size κ that introduces a set $A \subseteq \omega^\omega$ of size κ and a continuous pair coloring c_A on A such that (κ, c) and (A, c_A) are isomorphic in the sense that for some bijection $f : \kappa \to A$ we have $c(\alpha_1, \dots, \alpha_n) = c_A(f(\alpha_1), \dots, f(\alpha_n))$ for all $\{\alpha_1, \dots, \alpha_n\} \in [\kappa]^n$.

Notice that $\mathbb{P}_c$ forces c to be continuous not just with respect to some second countable topology on κ, but actually with respect to a separable metric topology.

Definition 2.1. For two functions s and t we write $s \perp t$ if $s \cup t$ is not a function. For a finite set $F = \{s_1, \dots, s_n\}$ of functions we write $\perp(F)$ or $\perp(s_1, \dots, s_n)$ if for each pair s, t of distinct elements of F we have $s \perp t$.

Definition 2.2. The conditions of $\mathbb{P}_c$ are of the form $p = (f_p, c_p)$ such that the following statements are satisfied:

(1) f_p is a finite partial function from κ to $\omega^{<\omega} \setminus \{\emptyset\}$ such that for all $\alpha, \beta \in \mathrm{dom}(p)$ with $\alpha \neq \beta$ we have $f_p(\alpha) \perp f_p(\beta)$.

(2) Let T_p be the tree generated by the elements of $\mathrm{ran}(f_p)$, i.e., the closure of $\mathrm{ran}(f_p)$ under initial segments. Then c_p is a partial coloring from $[T_p]^n$ to 2 with the following properties:

  (a) For all $s_1, \dots, s_n \in T_p$, if $c_p(s_1, \dots, s_n)$ is defined, then $\perp(s_1, \dots, s_n)$.

  (b) For all $s_1, \dots, s_n \in T_p$, if $c_p(s_1, \dots, s_n)$ is defined and $s'_1, \dots, s'_n \in T_p$ are such that $s_i \subseteq s'_i$ for all i, then $c_p(s'_1, \dots, s'_n)$ is defined and equals $c_p(s_1, \dots, s_n)$.

  (c) For all $\{\alpha_1, \dots, \alpha_n\} \in [\mathrm{dom}(f_p)]^n$, $c_p(f_p(\alpha_1), \dots, f_p(\alpha_n))$ is defined and equals $c(\alpha_1, \dots, \alpha_n)$.

For $p, q \in \mathbb{P}_c$ let $p \leq q$ if

(3) for all $\alpha \in \mathrm{dom}(f_q)$, $\alpha \in \mathrm{dom}(f_p)$ and $f_q(\alpha) \subseteq f_p(\alpha)$ and

(4) $c_p \restriction [T_q]^n = c_q$.

Lemma 2.3. *If c is continuous with respect to a second countable topology on κ, then $\mathbb{P}_c$ satisfies the countable chain condition.*

Proof. We fix a countable basis $\mathcal{B}$ for a Hausdorff topology on κ such that c is continuous with respect to the topology generated by $\mathcal{B}$.

Let $\mathcal{A} \subseteq \mathbb{P}_c$ be of size $\aleph_1$. We show that $\mathcal{A}$ contains $\aleph_1$ elements that are pairwise compatible.

After thinning out $\mathcal{A}$, we may assume that the domains of the f_p, $p \in \mathcal{A}$, all have the same size and form a Δ-system with root $r \subseteq \kappa$. Moreover, we may assume that for any two conditions $p, q \in \mathcal{A}$ we have $T_p = T_q$ and $c_p = c_q$. We may also assume that for all $p, q \in \mathcal{A}$, $f_p \restriction r$ and $f_q \restriction r$ are the same.

Now for every $p \in \mathcal{A}$ and every $\alpha \in \mathrm{dom}(f_p)$ we choose a set $O^p_\alpha \in \mathcal{B}$ such that $\alpha \in O^p_\alpha$, the sets O^p_α, $\alpha \in \mathrm{dom}(f_p)$, are pairwise disjoint and for all $\{\alpha_1, \dots, \alpha_n\} \in [\mathrm{dom}(f_p)]^n$, c is constant on $[O^p_{\alpha_1}, \dots, O^p_{\alpha_n}]$.

Since $\mathcal{B}$ is countable, we can thin out $\mathcal{A}$ further and assume that for all $p, q \in \mathcal{A}$ and all $\alpha \in \mathrm{dom}(f_p)$ and all $\beta \in \mathrm{dom}(f_q)$ with $f_p(\alpha) = f_q(\beta)$ we have $O^p_\alpha = O^q_\beta$.

We are finished if we can show

Claim 2.4. The conditions in $\mathcal{A}$ are pairwise compatible.

Let $p, q \in \mathcal{A}$ be distinct. We define a common extension s of p and q. For $\alpha \in r$ let $f_s(\alpha) = f_p(\alpha) = f_q(\alpha)$.

Now suppose $\alpha \in \mathrm{dom}(f_p) \setminus r$. Since $T_p = T_q$, there is a unique $\beta \in \mathrm{dom}(f_q)$ such that $f_q(\beta) = f_p(\alpha)$. Since $\alpha \notin r$, $\alpha \neq \beta$. Let $f_s(\alpha), f_s(\beta) \in \omega^{<\omega}$ be two extensions of $f_p(\alpha)$ such that $f_s(\alpha) \perp f_s(\beta)$.

This defines f_s and T_s. Clearly, (1) of Definition 2.2 is satisfied for f_s and, as far as (3) is concerned, the condition s is going to be a common extension of p and q.

We now define $c_s : [T_s]^n \to 2$. Let $F \in [T_s]^n$. If there is a family F' of pairwise incompatible restrictions of the elements of F such that $F' \in [T_p]^n$, then we have no choice and have to put $c_s(F) = c_p(F')$ in order to satisfy (2)(b) and (4) of Definition 2.2.

The difficulty here is (2)(c) of Definition 2.2. Given the family F there could be $\alpha_1, \dots, \alpha_n \in \mathrm{dom}(f_s)$ such that $F = f_s[\{\alpha_1, \dots, \alpha_n\}]$, but $\{\alpha_1, \dots, \alpha_n\}$ is neither a subset of $\mathrm{dom}(f_p)$ nor a subset of $\mathrm{dom}(f_q)$. By (2)(c) of Definition 2.2 we need to have $c(\alpha_1, \dots, \alpha_n) = c_p(F')$ if F' is as in the preceding paragraph. This is where the topology comes into play. For every i let $O_{\alpha_i} = O^p_{\alpha_i}$ if $\alpha_i \in \mathrm{dom}(f_p)$ and $O_{\alpha_i} = O^q_{\alpha_i}$ if $\alpha_i \in \mathrm{dom}(f_q)$.

Choose $\beta_1, \dots, \beta_n \in \mathrm{dom}(f_p)$ with $f_s(\alpha_i) = f_p(\beta_i)$ for every i. By the assumptions on $\mathcal{A}$, for every i we have $O_{\alpha_i} = O^p_{\beta_i}$. Since c is constant on $[O_{\beta_1}, \dots, O_{\beta_n}]$ and

$$\{\alpha_1, \dots, \alpha_n\} \in [O_{\alpha_1}, \dots, O_{\alpha_n}] = [O_{\beta_1}, \dots, O_{\beta_n}],$$

we have

$$c(\alpha_1, \dots, \alpha_n) = c(\beta_1, \dots, \beta_n) = c_p(F').$$

If there is no family F' as above, then (4) of Definition 2.2 will be satisfied if we define $c_s(F)$ only if F consists of maximal (with respect to $\subseteq$) elements of T_s, i.e., if $F \subseteq \mathrm{ran}(f_s)$. This also makes sure that (2)(a) of Definition 2.2 is satisfied. If F consists of maximal elements of T_s, then there is a unique set $\{\alpha_1, \dots, \alpha_n\} \in [\mathrm{dom}(f_s)]^n$ such that $F = f_s[\{\alpha_1, \dots, \alpha_n\}]$. We put $c_s(F) = c(\alpha_1, \dots, \alpha_n)$. This finishes the definition of c_s and it is easily checked that $s = (f_s, c_s)$ is indeed a common extension of p and q. This finishes the proof of the claim and hence of the lemma. □

Lemma 2.5. *For all $\alpha \in \kappa$ and all $m \in \omega$ the set*

$$D^m_\alpha = \{p \in \mathbb{P}_c : \alpha \in \mathrm{dom}(f_p) \wedge |f_p(\alpha)| > m\}$$

is dense in $\mathbb{P}_c$.

Proof. Let $\alpha \in \kappa$. We first consider the case $m = 0$. Let $p \in \mathbb{P}_c$. If $\alpha \in \mathrm{dom}(f_p)$, then $p \in D^0_\alpha$. If $\alpha \notin \mathrm{dom}(f_p)$, then let $k \in \omega$ be such that k is not the first coordinate of any sequence in $\mathrm{ran}(f_p)$. We define an extension $q \in D^0_\alpha$ of p.

Let $f_q(\alpha)$ be the sequence of length 1 with value k. For all $\beta \in \mathrm{dom}(f_p)$ let $f_q(\beta) = f_p(\beta)$. We have to put $c_q \restriction [T_p]^n = c_p$ and for every $F \in [\mathrm{dom}(f_q)]^n$ with $\alpha \in F$ we have to define $c_q(f_q[F]) = c(F)$, but this does not cause any conflicts with (2)(b) of Definition 2.2 since no non-empty initial segment of $f_q(\alpha)$ is an element of $T_p \setminus \{\emptyset\}$ and since the elements of $\mathrm{ran}(f_p)$ are pairwise incompatible.

Clearly, $q \in D^0_\alpha$ and $q \leq p$.

Now, if $p \in \mathbb{P}_c$, $m > 0$ and $\alpha \in \kappa$, then by the first part of this proof, we may assume that $\alpha \in \mathrm{dom}(f_p)$. In order to find a condition $q \leq p$ in D^m_α, we only have to extend $f_p(\alpha)$ to a sequence of length $> m$. We can then extend c_p according to (2) of Definition 2.2. This yields a condition $q \in D^n_\alpha$ with $q \leq p$. □

Now let $G \subseteq \mathbb{P}_c$ be a filter intersecting the sets D^m_α for all $\alpha \in \kappa$ and all $m \in \omega$. Then for every $\alpha \in \kappa$,

$$f(\alpha) = \bigcup \{f_p(\alpha) : p \in G \wedge \alpha \in \mathrm{dom}(f_p)\}$$

is an element of ω^ω. Thus, f is a function from κ to ω^ω. Note that f is actually 1-1.

For $F \in [\kappa]^n$ let $c_A(f[F]) = c(F)$.

Lemma 2.6. *c_A is continuous with respect to the subspace topology that A inherits from ω^ω.*

Proof. Let $F = \{\alpha_1, \dots, \alpha_n\} \in [\kappa]^n$. Then there is $p \in G$ such that $F \subseteq \mathrm{dom}(f_p)$. By (2) of Definition 2.2, $c_p(f_p[F])$ is defined and equals $c(F)$. Let $F' = \{\beta_1, \dots, \beta_n\} \in [\kappa]^n$ be such that $f_p(\alpha_i) \subseteq f(\beta_i)$ for all i.

Then there is some condition $q \leq p$ such that $q \in G$, $F' \subseteq \mathrm{dom}(f_q)$ and moreover, $|f_q(\beta_i)| \geq |f_p(\alpha_i)|$ for every i. Clearly, for every i, $f_p(\alpha_i) \subseteq f_q(\beta_i)$. Since $c_p \subseteq c_q$ and by (2)(b) of Definition 2.2,

$$c(F') = c_q(f_q[F']) = c_q(f_p[F]) = c_p(f_p[F]) = c(F).$$

It follows that $c_A(f[F]) = c_A(f[F'])$.

This shows that on $[A]^n$, $c_A(f[F])$ is already determined by sufficiently long finite initial segments of the elements of $f[F]$. In other words, c_A is indeed continuous. □

3. A characterization of potentially continuous colorings

Theorem 3.1. *A coloring $c : [\kappa]^n \to 2$ is potentially continuous iff $\mathbb{P}_c$ is c.c.c.*

Proof. If $\mathbb{P}_c$ is c.c.c., then c is potentially continuous since it is continuous in any $\mathbb{P}_c$-generic extension of the ground model by Lemma 2.6.

On the other hand, suppose that W is an extension of the ground model V such that c is continuous in W and $(\aleph_1)^V = (\aleph_1)^W$. Let us consider the forcing notion $\mathbb{P}_c$ in W. By Lemma 2.3, $\mathbb{P}_c$ is c.c.c. in W. But the conditions in $\mathbb{P}_c$ are absolute and hence, $(\mathbb{P}_c)^V$ is the same as $(\mathbb{P}_c)^W$. Now if $\mathcal{A} \subseteq \mathbb{P}_c$ is an uncountable antichain in V, then it is uncountable in W as well, because V and W have the same $\aleph_1$. It follows that $\mathbb{P}_c$ is c.c.c. in V. □

Theorem 3.1 easily implies

Corollary 3.2. *Assume* $\mathrm{MA}_{\aleph_1}$ *and* WOCA. *Then for every potentially continuous coloring $c : [\kappa]^2 \to 2$ on an uncountable cardinal κ there is an uncountable c-homogeneous set.*

Let us now analyze the proof of Lemma 2.3. Let κ be an uncountable cardinal and $c : [\kappa]^n \to 2$ a coloring. In the proof of Lemma 2.3, the continuity of the coloring c was used in the following way:

Let $\mathcal{A}$ be an uncountable family consisting of m-tuples $(\alpha_1, \dots, \alpha_m)$ of pairwise distinct ordinals. If c is continuous with respect to a second countable topology on κ, we may assume, after thinning out the family $\mathcal{A}$ if necessary, that there are disjoint open sets $O_1, \dots, O_m \subseteq \kappa$ such that $\mathcal{A} \subseteq O_1 \times \cdots \times O_m$ and for all $\{i_1, \dots, i_n\} \in [\{1, \dots, m\}]^n$, c is constant on $[O_{i_1}, \dots, O_{i_n}]$.

This is the crucial ingredient of the proof of Claim 2.4. It is clear that all that is really needed is that, given an uncountable family $\mathcal{A}$ of m-tuples with pairwise distinct entries, there are two distinct m-tuples $(\alpha_1, \dots, \alpha_m), (\beta_1, \dots, \beta_m) \in \mathcal{A}$ such that for every $\{i_1, \dots, i_n\} \in [\{1, \dots, m\}]^n$, c is constant on $[\{\alpha_{i_1}, \beta_{i_1}\}, \dots, \alpha_{i_n}, \beta_{i_n}\}]$.

This in fact gives a characterization of potentially continuous colorings.

Corollary 3.3. *A coloring $c : [\kappa]^n \to 2$ is potentially continuous if and only if for every uncountable family $\mathcal{A}$ of m-tuples consisting of pairwise distinct ordinals in κ there are two distinct tuples $(\alpha_1, \dots, \alpha_m), (\beta_1, \dots, \beta_m) \in \mathcal{A}$ such that for all $\{i_1, \dots, i_n\} \in [\{1, \dots, m\}]^n$, c is constant on $[\{\alpha_{i_1}, \beta_{i_1}\}, \dots, \{\alpha_{i_n}, \beta_{i_n}\}]$.*

Proof. The remark before the corollary together with the proof of Lemma 2.3 shows that $\mathbb{P}_c$ is c.c.c if the right hand side of the equivalence stated in the corollary holds.

On the other hand, if c is continuous, then the remark before the corollary shows that the right hand side of the equivalence holds. However, as with the c.c.c. of $\mathbb{P}_c$ in the proof of Theorem 3.1, the right hand side of the equivalence is downward absolute between models that agree on uncountability: if it holds in some extension W of V with $(\aleph_1)^V = (\aleph_1)^W$, then it holds in V as well. □

An alternative way to prove the implication from the right to the left in Corollary 3.3 is to construct an uncountable antichain in $\mathbb{P}_c$ directly from a counterexample to the condition of the right hand side of the equivalence, which is not hard.

It will be useful to have the following definition at our disposal:

Definition 3.4. For a coloring c as in Corollary 3.3 we say that two n-tuples $(\alpha_1, \dots, \alpha_m), (\beta_1, \dots, \beta_m)$ are of the same *type*, if for all $\{i_1, \dots, i_n\} \in [\{1, \dots, m\}]^n$

we have

$$c(\alpha_{i_1}, \dots, \alpha_{i_m}) = c(\beta_{i_1}, \dots, \beta_{i_m}).$$

Clearly, for fixed m there are only finitely many types of m-tuples. Hence, in the situation of Corollary 3.3 we may always assume that all m-tuples under consideration are of the same type.

A different formulation of Corollary 3.3 is

Corollary 3.5. *A coloring $c : [\kappa]^n \to 2$ is potentially continuous if and only if for every uncountable family $\mathcal{A}$ of m-tuples consisting of pairwise distinct ordinals in κ there are two distinct tuples $(\alpha_1^0, \dots, \alpha_m^0), (\alpha_1^1, \dots, \alpha_m^1) \in \mathcal{A}$ such that the m-tuples $(\alpha_1^{\varepsilon(1)}, \dots, \alpha_m^{\varepsilon(m)})$, $\varepsilon : \{1, \dots, m\} \to 2$, all have the same type.*

4. A potentially continuous coloring that is not continuous

Let us point out two colorings that are not potentially continuous. The most obvious example is Sierpinski's coloring. Let X be a set of reals of size $\aleph_1$ and choose a well-ordering $\prec$ on X. For $\{x, y\} \in [X]^2$ let $c_S(x, y) = 1$ if $\prec$ and the usual ordering $<$ on the reals agree on $\{x, y\}$ and let $c_S(x, y) = 0$ if the two orderings disagree.

Every c_S-homogeneous set is either an increasing or a decreasing sequence of reals, indexed by an ordinal. Since uncountable increasing or decreasing sequences of reals do not exist by the separability of $\mathbb{R}$, every c_S-homogeneous set is countable. This argument is sufficiently absolute to make sure that there is no way to add an uncountable c_S-homogeneous set to the set-theoretic universe. Since uncountable homogeneous sets for continuous colorings can be added by forcing, as was mentioned in the introduction, uncountable homogeneous sets can be added for potentially continuous colorings. It follows that c_S is not potentially continuous.

A more subtle example for a 2-coloring on $\aleph_1$ that is not potentially continuous is a coloring due to Todorčević [8, Section 8]. He defined a coloring $c_T : [\aleph_1]^2 \to 2$ with the following properties:

(1) No uncountable subset of $\aleph_1$ is c_T-homogeneous.
(2) An uncountable c_T-homogeneous set can be added by forcing, but not without killing a stationary subset of $\aleph_1$. In particular, no proper forcing adds an uncountable c_T-homogeneous set.

Observe that uncountable homogeneous sets for potentially continuous colorings can be added by proper forcing: first use some c.c.c. forcing to make the coloring continuous, then collapse $2^{\aleph_0}$ to $\aleph_1$ using some proper forcing and finally add an uncountable homogeneous set using some c.c.c. forcing. It follows that c_T is not potentially continuous.

It is a natural question whether there can be a potentially continuous coloring that is not already continuous. Note that a constant 2-coloring on a set X of size $> 2^{\aleph_0}$ is potentially continuous, it is enough to enlarge the continuum to $|X|$, but it is not continuous for the trivial reason that no second countable space is of size $> 2^{\aleph_0}$. However, consistently there are more interesting examples of potentially continuous that are not continuous.

Let $\mathbb{P}$ consist of all finite partial functions from $[\aleph_1]^2$ to 2 ordered by reverse inclusion. Clearly, $\mathbb{P}$ is isomorphic to the forcing notion that adds $\aleph_1$ Cohen reals. Let G be $\mathbb{P}$-generic over the ground model V. Now $c = \bigcup_{p \in G} p$ is a pair coloring on $\aleph_1$.

Lemma 4.1. *In $V[G]$, c is not continuous.*

Proof. Suppose in $V[G]$, τ is a second countable topology on $\aleph_1$ such that c is continuous with respect to τ. Fix a $\mathbb{P}$-name $\dot{c}$ for c. Since $\aleph_1$ is uncountable, it contains a nontrivial convergent sequence $(x_n)_{n\in\omega}$ with respect to τ. Now for every $y \in \aleph_1 \setminus \{x_n : n \in \omega\}$, the sequence $(c(x_n, y))_{n\in\omega}$ is eventually constant. We show that this cannot be the case.

Fix a $\mathbb{P}$-name $\dot{X}$ for the sequence $(x_n)_{n\in\omega}$. We may assume that $1_{\mathbb{P}}$ forces $\dot{X}$ to be a 1-1 sequence of countable ordinals. For every n, let $\dot{x}_n$ be a name for the n-th element of $\dot{X}$. We may assume that each $\dot{x}_n$ uses only countably many conditions from $\mathbb{P}$. Fix a countable set $C \subseteq \aleph_1$ such that all conditions used in any $\dot{x}_n$ have their domain contained in $[C]^2$. Let α be any countable ordinal outside C.

Now suppose that $p \in \mathbb{P}$ forces that $(\dot{c}(\dot{x}_n, \alpha))_{n\in\omega}$ is eventually constant with value $i \in 2$. Extending p if necessary, we may assume that p forces $(\dot{c}(\dot{x}_n, \alpha))_{n\in\omega}$ to be constant from some fixed $m \in \omega$ on.

Recursively choose conditions q_n, $n \in \omega$, such that each q_n is compatible with p, has its domain contained in $[C]^2$, decides $\dot{x}_{m+n}$ to be $\alpha_n \in \aleph_1$, and extends all q_k with $k < n$. Since the α_n have to be pairwise distinct, there is some $n \in \omega$ such that $\alpha_n \neq \alpha$ and $\{\alpha_n, \alpha\} \notin \operatorname{dom} p$. Since $\alpha \notin C$, $\{\alpha_n, \alpha\} \notin \operatorname{dom} q_n$. Now $r = p \cup q_n \cup \{(\{\alpha_n, \alpha\}, 1 - i)\}$ is a condition below p that forces $\dot{c}(\dot{x}_{m+n}, \alpha)$ to be $1 - i$, contradicting the choice of p and m. □

Lemma 4.2. *In $V[G]$, there is no uncountable c-homogeneous set.*

Proof. Suppose there is an uncountable c-homogeneous set H. Fix a $\mathbb{P}$-name $\dot{H}$ for H. We may assume that $\dot{H}$ is forced by $1_{\mathbb{P}}$ to be an uncountable subset of $\aleph_1$. Suppose $p \in \mathbb{P}$ forces that H is homogeneous of color $i \in 2$.

For every $\alpha \in \aleph_1$ choose a condition $p_\alpha \leq p$ and an ordinal $\beta_\alpha \in \aleph_1$ such that p_α forces that the α-th element of $\dot{H}$ is β_α. For each α consider the set $d_\alpha = \{\beta_\alpha\} \cup \bigcup \operatorname{dom} p_\alpha$, i.e., the set of all ordinals that appear in the domain of p_α together with β_α. Let $S \subseteq \aleph_1$ be uncountable such that the d_α, $\alpha \in S$, form a Δ-system with root r. After thinning out S we may assume that the p_α, $\alpha \in S$, are pairwise compatible. Now the ordinals β_α, $\alpha \in S$, are pairwise distinct. We may further assume that no β_α, $\alpha \in S$, is an element of r.

Let α and α' be two distinct elements of S. By the choice of S, $\{\beta_\alpha, \beta_{\alpha'}\} \notin \operatorname{dom} p_\alpha \cup \operatorname{dom} p_{\alpha'}$. Hence $q = p_\alpha \cup p_{\alpha'} \cup \{(\{\beta_\alpha, \beta_{\alpha'}\}, 1 - i)\}$ is a condition below p such that

$$q \Vdash \beta_\alpha, \beta_{\alpha'} \in \dot{H} \wedge \dot{c}(\beta_\alpha, \beta_{\alpha'}) = 1 - i,$$

contradicting the choice of p. □

Lemma 4.3. *In $V[G]$, c is potentially continuous.*

Proof. We use the characterization from Corollary 3.3. In $V[G]$, let $\mathcal{A}$ be an uncountable set of m-tuples of pairwise distinct ordinals in $\aleph_1$. We may assume that the m-tuples from $\mathcal{A}$ all have the same type. We argue that is can also be assumed that the tuples in $\mathcal{A}$ are pairwise disjoint.

First we thin out $\mathcal{A}$ in such a way that the sets of entries of the tuples in $\mathcal{A}$ form a Δ-system with some root w. We may assume that every element of the root appears on the same coordinate in every tuple from $\mathcal{A}$. Now we delete all the coordinates in the m-tuples from $\mathcal{A}$ that have values in the root of the Δ-system.

It is easily checked that the modified family $\mathcal{A}$ satisfies the condition on the right hand side of the equivalence in Corollary 3.3 if and only if the original family does.

Now let $\dot{\mathcal{A}}$ be a name for $\mathcal{A}$ and assume that $p \in \mathbb{P}$ forces $\dot{\mathcal{A}}$ to be an uncountable set of pairwise disjoint m-tuples of pairwise distinct countable ordinals such that all m-tuples from $\mathcal{A}$ have the same type. Let $\dot{X}$ be a name for a 1-1 enumeration of $\mathcal{A}$ indexed by $\aleph_1$ and for each $\alpha \in \aleph_1$ let $\dot{x}_\alpha$ be a name for $\dot{X}(\alpha)$.

For each $\alpha \in \aleph_1$ choose a condition $p_\alpha \leq p$ that decides $\dot{x}_\alpha$ to be $(\beta_{1,\alpha}, \dots, \beta_{m,\alpha})$. For each α let $d_\alpha = \{\beta_{1,\alpha}, \dots, \beta_{m,\alpha}\} \cup \bigcup \operatorname{dom} p_\alpha$. Let $S \subseteq \aleph_1$ be uncountable such that the d_α, $\alpha \in S$, form a Δ-system with root r and such that the conditions p_α, $\alpha \in S$, are pairwise compatible.

Since the $\dot{x}_\alpha$ are forced to be pairwise disjoint and the p_α, $\alpha \in S$, are pairwise compatible, the $(\beta_{1,\alpha}, \dots, \beta_{m,\alpha})$, $\alpha \in S$, are pairwise disjoint. It follows that there are two distinct ordinals $\alpha, \alpha' \in S$ such that the corresponding m-tuples $(\beta_{1,\alpha}, \dots, \beta_{m,\alpha})$ and $(\beta_{1,\alpha'}, \dots, \beta_{m,\alpha'})$ are disjoint from r.

Since p_α and $p_{\alpha'}$ are compatible, $q = p_\alpha \cup p_{\alpha'}$ is a condition in $\mathbb{P}$. Note that for all $\{i,j\} \in [\{1,\dots,m\}]^2$, $\{\beta_{i,\alpha}, \beta_{j,\alpha'}\} \notin \operatorname{dom}(q)$. Hence we can extend q to a condition q' that forces that c is constant on every set $[\{\beta_{i,\alpha}, \beta_{i,\alpha'}\}, \{\beta_{j,\alpha}, \beta_{j,\alpha'}\}]$, $\{i,j\} \in [\{1,\dots,m\}]^2$.

Clearly, this argument in fact shows that conditions q' as above are dense below p. It follows that G contains such a condition q'. Hence $\mathcal{A}$ contains two distinct m-tuples $(\beta_1, \dots, \beta_m)$ and $(\beta'_1, \dots, \beta'_m)$ such that for all $\{i,j\} \in [\{1,\dots,m\}]^2$, c is constant on $[\{\beta_i, \beta'_i\}, \{\beta_j, \beta'_j\}]$.

By Lemma 3.3, this implies that c is potentially continuous. $\square$

5. Destroying potential continuity

In this section we consider once more the coloring c from Section 4 that has been defined from $\aleph_1$ Cohen reals over any ground model. We show that there is some c.c.c. forcing that forces that c is not potentially continuous.

Definition 5.1. Let $\mathbb{Q}$ consist of conditions of the form

$$\{\{\alpha_1, \alpha_1 + 1\}, \dots, \{\alpha_k, \alpha_k + 1\}\}$$

such that for all $i \in \{1, \dots, k\}$,

(1) $\alpha_i \in \aleph_1$,
(2) α_i is even,
(3) $\alpha_i + 1 < \alpha_{i+1}$,
(4) $c(\alpha_i, \alpha_i + 1) = 0$,
(5) for all $j \neq i$, $c(\alpha_i, \alpha_j + 1) = 1$.

$\mathbb{Q}$ is ordered by reverse inclusion.

In other words, $\mathbb{Q}$ consists of finite approximations of a very simple counterexample to the right hand side of the equivalence in Corollary 3.3.

For what follows, let $\dot{\mathbb{Q}}$ be a $\mathbb{P}$-name for $\mathbb{Q}$.

Lemma 5.2. *For each $\alpha \in \aleph_1$ let*

$$D_\alpha = \{q \in \mathbb{Q} : \exists \beta \geq \alpha (\{\beta, \beta + 1\} \in q)\}.$$

Then every D_α is dense in $\mathbb{Q}$.

Proof. Let $q \in \mathbb{Q}$. Choose a $\mathbb{P}$-name $\dot{q}$ for q. Let $p \in \mathbb{P}$ be such that $p \Vdash \dot{q} \in \dot{\mathbb{Q}}$. Let $\beta \in \aleph_1$ be such that

(i) $\beta \geq \alpha$,
(ii) β is even,
(iii) $\beta, \beta + 1 \notin \bigcup \operatorname{dom}(p)$.

Now we can easily extend p to a condition p' such that p' forces that $\dot{q} \cap \{\beta, \beta + 1\}$ is a condition in $\mathbb{Q}$. $\square$

Now it is clear that if G is $\mathbb{P}$-generic over the ground model V, then forcing with $\mathbb{Q}$ over $V[G]$ either collapses $(\aleph_1)^{V[G]}$ or introduces an uncountable set of pairs in $\aleph_1$ that contradicts the right hand side of the equivalence in Corollary 3.3.

Thus, in order to show that $\mathbb{Q}$ destroys the potential continuity of c, it remains to show that forcing with $\mathbb{Q}$ does not collapse $\aleph_1$. To this end, we show that $\mathbb{P} * \dot{\mathbb{Q}}$ is c.c.c.

Lemma 5.3. *$\mathbb{P} * \dot{\mathbb{Q}}$ is c.c.c.*

Proof. Let $(p_\nu, \dot{q}_\nu)_{\nu\in\aleph_1}$ be a family of conditions in $\mathbb{P} * \dot{\mathbb{Q}}$. After shrinking the p_ν if necessary, we may assume that for each α, p_ν decides $\dot{q}_\nu$. We therefore drop the dot on $\dot{q}_\nu$ and consider each q_ν as a set of the form

$$\{\{\alpha_1, \alpha_1 + 1\}, \dots, \{\alpha_k, \alpha_k + 1\}\}$$

such that for all $i \in \{1, \dots, k\}$ the conditions (1)–(3) of Definition 5.1 are satisfied. For each $\nu \in \aleph_1$ let $d_\nu = \bigcup q_\nu \cup \bigcup \operatorname{dom} p_\nu$, i.e., let d_ν consist of the finitely many countable ordinals involved in forming (p_ν, q_ν).

We uniformize the family $(p_\nu, q_\nu)_{\nu\in\aleph_1}$ as much as possible. After thinning out and reindexing the family we may assume

(i) the d_ν form a Δ-system with root d and and for every ν, d is an initial part of d_ν,
(ii) the p_ν are pairwise compatible and
(iii) for all $\nu, \nu' \in \aleph_1$, d_ν and $d_{\nu'}$ are of the same size and the order preserving map between d_ν and $d_{\nu'}$ is an isomorphism between the structures $(d_\nu, d, q_\nu, p_\nu^{-1}(1))$ and $(d_{\nu'}, d, q_{\nu'}, p_{\nu'}^{-1}(1))$ where q_ι and $p_\iota^{-1}(1)$ are considered as binary symmetric relations on d_ι, $\iota \in \{\nu, \nu'\}$.

Claim 5.4. For $\nu, \nu' \in \aleph_1$, the conditions (p_ν, q_ν) and $(p_{\nu'}, q_{\nu'})$ are compatible in $\mathbb{P} * \dot{\mathbb{Q}}$.

Assume that $\nu \neq \nu'$. Since p_ν and $p_{\nu'}$ are compatible, $p = p_\nu \cup p_{\nu'}$ is a condition in $\mathbb{P}$. Let $q = q_\nu \cup q_{\nu'}$. Since p_ι forces that q_ι satisfies condition (4) in Definition 5.1 for $\iota \in \{\nu, \nu'\}$, p forces that this condition is satisfied for q as well.

It may be the case that p does not force that q satisfies condition (5) of Definition 5.1. We have to show that p can be extended to a condition $r \in \mathbb{P}$ that forces that q satisfies (5). In this case (r, q) is a common extension of (p_ν, q_ν) and $(p_{\nu'}, q_{\nu'})$.

We write q as

$$\{\{\alpha_1, \alpha_1 + 1\}, \dots, \{\alpha_k, \alpha_k + 1\}\}.$$

Let $i, j \in \{1, \dots, k\}$ be such that $i \neq j$. It is now sufficient to show that it is not the case that $\{\alpha_i, \alpha_j + 1\} \in \operatorname{dom} p$ and p forces that $c(\alpha_i, \alpha_j + 1) = 0$, i.e., $p(\alpha_i, \alpha_j + 1) = 0$.

Assume $\{\alpha_i, \alpha_j + 1\} \in \operatorname{dom} p$, say $\{\alpha_i, \alpha_j + 1\} \in \operatorname{dom} p_\nu$.

If $\{\alpha_i, \alpha_i + 1\} \notin q_\nu$, then $\{\alpha_i, \alpha_i + 1\} \in q_{\nu'}$. It follows that $\alpha_i \in d_\nu \cap d_{\nu'} = d$. But now by (iii),

$$\{\alpha_i, \alpha_i + 1\} \in q_\nu \Leftrightarrow \{\alpha_i, \alpha_i + 1\} \in q_{\nu'},$$

a contradiction. Hence $\{\alpha_i, \alpha_i + 1\} \in q_\nu$. By the same argument, $\{\alpha_j, \alpha_j + 1\} \in q_\nu$.

But if $\{\alpha_i, \alpha_i + 1\}, \{\alpha_j, \alpha_j + 1\} \in q_\nu$, then p_ν, and hence p, forces that $c(\{\alpha_i, \alpha_j + 1\}) = 1$. This finishes the proof of the claim and hence of the lemma. □

We now easily get the following:

Theorem 5.5. *Let G be $\mathbb{P}$-generic over the ground model V. Then in $V[G]$ there are a potentially continuous coloring c and a c.c.c. forcing $\mathbb{Q}$ such that forcing with $\mathbb{Q}$ destroys the potential continuity of c. In particular, the forcing notion $\mathbb{P}_c \times \mathbb{Q}$ is not c.c.c.*

Proof. We argue in $V[G]$. Let c be the generic coloring added by $\mathbb{P}$. From Lemma 5.3 it follows that $\mathbb{Q}$ is c.c.c. By the remark following Lemma 5.2, forcing with $\mathbb{Q}$ destroys the potential continuity of c.

Now consider the forcing notion $\mathbb{P}_c \times \mathbb{Q}$. By the absoluteness of the conditions of $\mathbb{P}_c$ this product is equivalent to the iteration $\mathbb{Q} * \dot{\mathbb{P}}_c$ where $\dot{\mathbb{P}}_c$ is a $\mathbb{Q}$-name for $\mathbb{P}_c$. Since $\mathbb{Q}$ destroys the potential continuity of c, it forces that $\mathbb{P}_c$ is not c.c.c. Hence $\mathbb{Q} * \dot{\mathbb{P}}_c$, and therefore $\mathbb{Q} \times \mathbb{P}_c$, fails to be c.c.c. □

6. Problems

Let us conclude with two open problems. The first question was asked by Veličković [9].

Question 6.1. Is it consistent that there are two potentially continuous colorings c and c' such that $\mathbb{P}_c \times \mathbb{P}_{c'}$ is not c.c.c.? In other words, can there be two potentially continuous coloring that cannot be made continuous at the same time?

Question 6.2. Can every coloring that is not continuous be forced to be not even potentially continuous?

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