FUNCTIONS WITH MANY LOCAL EXTREMA

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1. Introduction

In [1] the question was studied whether a continuous function f from a topological space X into the real line can have a local extremum at every point of X without being constant. Among other things it was observed if X is a space of weight $<|\mathbb{R}|$, then a continuous function $f : X \to \mathbb{R}$ that has a local extremum at every point of X is constant. Also, if X is a connected linear order in which every family of pairwise disjoint open intervals is of size $<|\mathbb{R}|$ and $f : X \to \mathbb{R}$ is continuous and has a local extremum at every point of X, then f is constant.

The proof of the latter fact given in [1] shows that if X is a connected linear order and $f : X \to \mathbb{R}$ is continuous and has a local extremum at every point of X, then f is constant on a nonempty open interval. In fact, the collection of open intervals on which f is constant has a dense union.

In this note, we investigate the question how many local extrema a non-constant continuous function into the reals can have. We restrict our attention to functions defined on the unit interval $[0, 1]$.

It is relatively easy to construct a continuous function $f : [0, 1] \to \mathbb{R}$ that is not constant and whose set of local minima is open and dense. Just choose a closed nowhere dense set $A \subseteq [0, 1]$ of positive measure (see Lemma 1) and let $f(x)$ be the measure of $A \cap [0, x]$. Then clearly, f is continuous, not constant and constant on every open interval disjoint from A. In particular, f has a local minimum and maximum at every point of $X \setminus A$.

This example shows that we should consider functions that are not constant on any nonempty open interval.

2. Measure

The following lemma is well-known.

Lemma 1. *Let $\varepsilon > 0$. Then there is a closed, nowhere dense set $A \subseteq [0, 1]$ of measure at least $1 - \varepsilon$.*

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Proof. Let $\{(a_n, b_n) : n \in \mathbb{N}\}$ be the collection of all open subintervals of $[0,1]$ with rational endpoints. For each $n \in \mathbb{N}$ let $(c_n, d_n) \subseteq (a_n, b_n)$ be an open interval of length at most $2^{-n} \cdot \varepsilon$. Now $B = \bigcup_{n \in \mathbb{N}} (c_n, d_n)$ is a dense open set of measure at most ε. Hence, the set $A = [0,1] \setminus B$ is closed, nowhere dense and of measure at least $1 - \varepsilon$. □

Theorem 2. *For every $\varepsilon > 0$ there is a continuous function $f : [0,1] \to \mathbb{R}$ such that f is not constant on any non-empty open interval and the set of local minima of f is dense in $[0,1]$ and of measure at least $1 - \varepsilon$.*

Proof. Let $A \subseteq [0,1]$ be a closed, nowhere dense subset of $[0,1]$ of measure at least $1 - \varepsilon$ as in Lemma 1. Let $C([0,1])$ denote the Banach space of all continuous real valued functions on $[0,1]$ with the supremum norm. Let X denote the closed subset of $C([0,1])$ of all non-negative functions that are zero on A.

Then X is a complete metric space. Let $a, b \in [0,1]$ be such that $a < b$. We say that a function $f \in X$ has a *proper local minimum on* (a,b) if there are $x_1, x_2, x_3 \in (a,b)$ such that $x_1 < x_2 < x_3$ and $f(x_2) < f(x_1), f(x_3)$. Note that if f has a proper local minimum on (a,b) and this is witnessed by x_1, x_2, x_3, then f actually has a local minimum on (x_1, x_3). Since A is nowhere dense, the set $X_{a,b}$ of all $f \in X$ that have a proper local minimum on (a,b) is open and dense in X. By the Baire Category Theorem, there is a function f that has a proper local minimum, and hence a local minimum, on every interval $(a,b) \subseteq [0,1]$ with rational endpoints. This function f works for the theorem. □

3. Category

We point out that the analog of Theorem 2 for category fails.

Theorem 3. *If $f : [0,1]\mathbb{R}$ is continuous and not constant on any non-empty open interval, then the set of local minima of f is meager.*

The proof of this theorem uses the following lemma.

Lemma 4. *The set of local minima of a continuous function $f : [0,1] \to \mathbb{R}$ is F_σ.*

Proof. For $a, b, c, d \in [0,1] \cap \mathbb{Q}$ with $a < b < c < d$ consider the set

$$M_{a,b,c,d} = \{x \in [b,c] : f(x) = \min f[(a,d)]\}.$$

Clearly, $M_{a,b,c,d}$ is closed and every element of $M_{a,b,c,d}$ is a local minimum of f. On the other hand, if x is a local minimum of f, then there are $a, b, c, d \in [0,1] \cap \mathbb{Q}$ such that $a < b < c < d$ and $x \in M_{a,b,c,d}$. It follows that the set of local minima of f is equal to

$$\bigcup \{M_{a,b,c,d} : a, b, c, d \in [0,1] \cap \mathbb{Q} \wedge a < b < c < d\},$$

which is clearly F_σ. □

Proof of Theorem 3. By Lemma 4, the set M of local minima of f can be written as $\bigcup_{n\in\mathbb{N}} M_n$ where each M_n is closed. Assume that M is not meager. Then for some $n \in \mathbb{N}$, M_n is somewhere dense. Since M_n is closed, M_n actually contains a non-empty open interval (a, b). But a continuous function that has a local minimum at each point of a nonempty interval is constant on that interval. A contradiction. □

Corollary 5. *If $f : [0, 1] \to \mathbb{R}$ is not constant on any non-empty open interval, then the set of local extrema of f is meager. However, the set can be dense and of positive measure.*

References

[1] E. Behrends, S. Geschke, T. Natkaniec, *Functions for which all points are a local minimum or maximum*, submitted

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