PREPARATION FOR THE FINAL EXAM MATH 170, FALL 2007

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Problem 1. Show that there is a positive real number a such that $a^2 = 3$. In other words, show that $\sqrt{3}$ exists.

Proof. Consider the function $f(x) = x^2$. Clearly, $f(1) = 1$, $f(2) = 4$ and f is continuous. Hence, by the Intermediate Value Theorem, there is a number $a \in [1, 2]$ such that $a^2 = f(a) = 3$. $\square$

Problem 2. Show that

$$\lim_{x \to 0^+} \frac{1}{x^2} = \infty.$$

Proof. Let $M > 0$. Let $\varepsilon = 1/\sqrt{M}$. If $x > 0$ and $x < \varepsilon$, then $\frac{1}{x^2} > \frac{1}{\varepsilon^2} = M$. This shows that for every $M > 0$ there is $\varepsilon > 0$ such that for all $x > 0$ with $x < \varepsilon$ we have $\frac{1}{x^2} > M$. $\square$

Problem 3. Show, using the definition of the derivative as a limit, that for every $x \in \mathbb{R}$ we have $(2x^3 + 1)' = 6x^2$.

Proof.

$$(2x^3 + 1)' = \lim_{h \to 0} \frac{(2(x + h)^3 + 1) - (2x^3 + 1)}{h} =$$

$$\lim_{h \to 0} \frac{2(x^3 + 3x^2h + 3xh^2 + h^3) + 1 - 2x^3 - 1}{h} =$$

$$\lim_{h \to 0} \frac{2x^3 + 6x^2h + 6xh^2 + 2h^3 - 2x^3}{h} =$$

$$\lim_{h \to 0} \frac{6x^2h + 6xh^2 + 2h^3}{h} = \lim_{h \to 0} (6x^2 + 6xh + h^2) = 6x^2$$

$\square$

Problem 4. Show that for every real number c the equation $x^4 + 4x + c = 0$ has at most two real roots.

Proof. Let c be any real number. Let $f(x) = x^4 + 4x + c$. This function is continuous and differentiable. Therefore Rolle's Theorem applies, i.e., between any two zeros of f there is a zero of f'. It follows that f can have at most one zero more than f'.

But $f'(x) = 4x^3 + 4$. If $f'(x) = 0$, then $4x^3 = -4$ and hence $x = -1$. It follows that f' has only a single zero. It follows that f cannot have more than 2 zeros. $\square$

Problem 5. Show that two antiderivatives of the same function only differ by a constant.

Proof. Let F and G be antiderivatives of f. Then for every x, $(F-G)'(x) = F'(x) - G'(x) = f(x) - f(x) = 0$. Now assume that $F - G$ was not constant. Then there are x_1 and x_2 such that $x_1 \neq x_2$ and $(F-G)(x_1) \neq (F-G)(x_2)$. By the Mean Value Theorem there is some x such that

$$(F-G)'(x) = \frac{(F-G)(x_2) - (F-G)(x_1)}{x_2 - x_1} \neq 0,$$

contradicting the fact that $(F-G)'(x) = 0$ for every x. It follows that $F - G$ is constant. □

Problem 6.

a) $$\lim_{x\to\infty} \frac{x^4 - 7x + 1}{3x^4 + 10000x} = \lim_{x\to\infty} \frac{1 - \frac{7}{x^3} + \frac{1}{x^4}}{3 + \frac{10000}{x^3}} = \frac{1}{3}$$

b) $$\lim_{x\to 0}(\sin x \cdot \ln x) = \lim_{x\to 0} \frac{\ln x}{\frac{1}{\sin x}} = \lim_{x\to 0} \frac{\frac{1}{x}}{-\frac{\cos x}{\sin^2 x}} =$$
$$\lim_{x\to 0} \frac{-\sin^2 x}{x \cos x} = \lim_{x\to 0} \frac{-2\sin x \cos x}{\cos x - x \sin x} = \frac{-2 \cdot 0 \cdot 1}{1 - 0 \cdot 0} = 0$$

Problem 7.

a) $$f'(x) = (\cosh(\arcsin x))' = \sinh(\arcsin x) \cdot (\arcsin x)' = \frac{\sinh(\arcsin x)}{\sqrt{1-x^2}}$$

b) $$f'(x) = (x^{\ln x})' = (e^{\ln(x^{\ln x})})' = (e^{(\ln x)^2})' =$$
$$e^{(\ln x)^2} \cdot ((\ln x)^2)' = e^{(\ln x)^2} \cdot 2 \cdot \ln x \cdot \frac{1}{x} = 2 \cdot x^{\ln x - 1} \cdot \ln x$$

Problem 8.

a) $$\int (e^{3x} + x^2)dx = \frac{1}{3}e^{3x} + \frac{1}{3}x^3 + C$$

b) We have
$$\int \tan x dx = \int \frac{\sin x}{\cos x} dx.$$

Now substitute $u = \cos x$. This yields $\frac{du}{dx} = -\sin x$ and thus $\sin x dx = -du$. The integral becomes

$$-\int \frac{1}{u} du = -\ln u + C = -\ln \cos x + C.$$

Problem 9.

a) $$\int_0^1 2xe^{x^2} dx = \left[e^{x^2}\right]_0^1 = e - 1$$

b) We have
$$\int_0^\pi \left(\frac{3}{x} + \cos x\right) dx = [3\ln x + \sin x]_0^\pi = 3\ln \pi + \sin \pi - 3\ln 0 - \sin 0,$$

but $\ln 0$ is undefined and hence we cannot compute this integral. (Sorry for this!)

Problem 10. Let f be a continuous function defined of the interval $[a, b]$. Write down the formal definition of the real number $\int_a^b f(x)dx$.

Answer. For each integer $n > 0$ and every $i \in \{1, \ldots, n\}$ let $\Delta x_n = \frac{b-a}{n}$ and $x_{n,i} = a + i \cdot \Delta x_n$. Now

$$\int_a^b f(x)dx = \lim_{n\to\infty} \left(\sum_{i=1}^{n} f(x_{n,i}) \cdot \Delta x_n \right)$$

□

Problem 11. State the Fundamental Theorem of Calculus in one of its two forms. The book calls the two forms Part 1 and Part 2, but really they are the same theorem.

Answer. Part 1: Let f be a continuous function on the closed interval $[a, b]$. Then the function $g(x)$ defined on $[a, b]$ by

$$g(x) = \int_a^x f(t)dt$$

is an antiderivative of f on (a, b).

Part 2: Let f be a continuous function on the closed interval $[a, b]$ and let F be an antiderivative of f. Then

$$\int_a^b f(x)dx = F(b) - F(a).$$

□

Problem 12. The speed of a car at time $t \in [0, 1]$ is given by $f(t) = -90t^2 + 90t$. The time is given in hours, the speed in miles per hour. What is the maximal speed? What is the distance traveled at time $t = 1$?

Answer. The distance traveled (in miles) is

$$\int_0^1 f(t)dt = \int_0^1 (-90t^2 + 90t)dt = [-30t^3 + 45t^2]_0^1 = 15.$$

In order to find the maxima of $f(t)$, we calculate the derivative.

$$f'(t) = -180t + 90$$

Therefore $f'(t) = 0$ if and only if $t = 1/2$. Clearly this is a global maximum. The maximal speed (in miles per hour) is

$$f(1/2) = -90 \cdot \frac{1}{4} + 90 \cdot \frac{1}{2} = 22.5.$$

□