1.1 Introduction to vectors
1.2 Lengths and dot products

January 28th, 2013
Math 301
Notation for linear systems

\[ 12w + 4x + 23y + 9z = 0 \]
\[ 2u + v + 5w - 2x + 2y + 8z = 1 \]
\[ -5u + v - 6w + 2x + 4y - z = 6 \]
\[ 8u - 4v - 5w - x - 7y = 7 \]
\[ 11u + 3v + 9x + y + 9z = 11 \]
\[ 3u - 2v - 8w - 15x + 5y - 6z = 45 \]

It is going to get rather cumbersome to always have to use \( u, v, w \) or \( x, y, z, \ldots \) when we talk about larger systems.
Vectors

Examples of a *vector* in two dimensions:

\[
\mathbf{v} = \begin{bmatrix} 2 \\ -5 \end{bmatrix} \quad \text{or} \quad \mathbf{v} = \begin{bmatrix} 3 \\ 1 \end{bmatrix} \quad \text{or} \quad \mathbf{v} = (4, 5)
\]

or in three dimensions

\[
\mathbf{v} = \begin{bmatrix} 2 \\ 5 \\ -1 \end{bmatrix} \quad \mathbf{v} = \begin{bmatrix} 4 & 2 & 3 \end{bmatrix} \quad \mathbf{v} = (1, 2, 3)
\]

“bold face” type is used for vector variable names
Vectors

Entries of a general vector are denoted using subscripts:
\[ \mathbf{v} = (v_1, v_2, v_3) \]

or for a vector \( \mathbf{v} \) in \( \mathbb{R}^n \), we write the entries as
\[ \mathbf{v} = (v_1, v_2, \ldots, v_i, \ldots, v_n) \]

Notice that the vector name is bold-face, but the scalar entries are just italicized.

We might still use \((x, y, z)\) to denote a vector, but in general it is more convenient to use subscripts.
Vector addition is easy. If we have vectors $\mathbf{v}$ and $\mathbf{w}$ defined as

$$\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \quad \text{and} \quad \mathbf{w} = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$$

then $\mathbf{v} + \mathbf{w}$ is defined as

$$\mathbf{v} + \mathbf{w} = \begin{bmatrix} v_1 + w_1 \\ v_2 + w_2 \end{bmatrix}$$

Simply add the entries component-wise to create a new vector.
We can also multiply \( \mathbf{v} \) by a scalar \( c \) to get a new vector

\[
    c\mathbf{v} = \begin{bmatrix}
        c v_1 \\
        c v_2
    \end{bmatrix}
\]

Examples:

\( \mathbf{v} = (4, 2, 7) \) and \( \mathbf{w} = (-1, -1, 3) \).

Then

\[
    \mathbf{v} + \mathbf{w} = (3, 1, 10), \quad 3\mathbf{v} = (12, 6, 21)
\]
Linear combinations

Given vectors \( \mathbf{v} \) and \( \mathbf{w} \), and scalars \( a \) and \( b \), we can form a linear combination to get a new vector:

\[
 a \mathbf{v} + b \mathbf{w} = \begin{bmatrix} a v_1 + b w_1 \\ a v_2 + b w_2 \end{bmatrix}
\]

Show that the following can be expressed as linear combinations:

(a) Vector addition? \( a = b = 1 \)
(b) Vector subtraction? \( a = 1, \ b = -1 \)
(c) The zero vector? \( a = b = 0 \)
(d) Scalar multiplication? \( a = c, \ b = 0 \)
We can plot vectors $\mathbf{v} = (2, 4)$ and $\mathbf{w} = (-1, 3)$ as points in the plane.

Or we can plot them as vectors.
Vector addition

Vector $\mathbf{v} + \mathbf{w} = (1, 7)$, which we can picture geometrically by first translating $\mathbf{v}$ to the head of $\mathbf{w}$ and drawing the sum.

All parallel vectors of the same length are the same vector!
Or we can translate $\mathbf{w}$ to the head of $\mathbf{v}$.

All parallel vectors of the same length are the same vector!
To subtract $\mathbf{w}$ from $\mathbf{v}$, we first negate $\mathbf{w}$ and then add the negated vector to $\mathbf{v}$.

Check:

$$\mathbf{w} + (\mathbf{v} - \mathbf{w}) = \mathbf{v}$$

$v - w = (3, 1)$
Do we need to be at the origin?

\[
\mathbf{v} = (-7, 2) \\
\mathbf{w} = (5, 3) \\
\mathbf{v} + \mathbf{w} = (-2, 5)
\]
Do we need to be at the origin?

\[ \mathbf{v} = (-7, 2) \]
\[ \mathbf{w} = (5, 3) \]
\[ \mathbf{v} - \mathbf{w} = (-12, -1) \]
Scalar multiplication

Multiplication by a scalar simply “scales” the length of the vector.
Given \( \mathbf{v} \) and \( \mathbf{w} \), what are the possible linear combinations?

\[
\mathbf{u} = a \mathbf{v} + b \mathbf{w}
\]

Can we get any point in the plane this way?

Will this work for any two vectors \( \mathbf{v} \) and \( \mathbf{w} \)?
Linear combinations

Given two vectors in 2d, can we determine if they fill only a line or the entire plane?

Will \( u = \begin{bmatrix} 3 \\ 4 \end{bmatrix} \) and \( v = \begin{bmatrix} -3 \\ -4 \end{bmatrix} \) fill the plane?

No - they only fill a line because the vectors are scalar multiples of each other.

What about \( u = \begin{bmatrix} 3 \\ 4 \end{bmatrix} \) and \( v = \begin{bmatrix} -3 \\ -1 \end{bmatrix} \)?

Yes - linear combinations of \( u \) and \( v \) fill the plane.
In 3d?

Do linear combinations of \( \mathbf{u}, \mathbf{v} \) and \( \mathbf{w} \) fill a line, a plane or all of three space?

\[
\mathbf{u} = \begin{bmatrix} 1 \\ 3 \\ -2 \end{bmatrix}, \quad \mathbf{v} = \begin{bmatrix} 2 \\ -1 \\ 4 \end{bmatrix} \quad \text{and} \quad \mathbf{w} = \begin{bmatrix} 3 \\ 2 \\ 2 \end{bmatrix}
\]

A plane, since the last vector is a linear combination of the first two.
The “dot product” is defined as

\[ \mathbf{v} \cdot \mathbf{w} = v_1 w_1 + v_2 w_2 + v_3 w_3, \quad \text{in} \quad \mathbb{R}^3 \]

Example: \( \mathbf{v} = (2, 3) \) and \( \mathbf{w} = (-4, 1) \). Then

\[ \mathbf{v} \cdot \mathbf{w} = (2)(-4) + (3)(1) = -5 \]

Example: \( \mathbf{v} = (1, 3, 7) \) and \( \mathbf{w} = (2, 0, -2) \). Then

\[ \mathbf{v} \cdot \mathbf{w} = (1)(2) + (3)(0) + (7)(-2) = -12 \]
Dot product

Sometimes called the “inner product”, or a “scalar product” or the Cartesian product.

\[ \mathbf{v} \cdot \mathbf{w} = \mathbf{w} \cdot \mathbf{v} \]

Rules :

\[ (c\mathbf{v}) \cdot \mathbf{w} = c(\mathbf{v} \cdot \mathbf{w}) \]

\[ \mathbf{u} \cdot (\mathbf{v} + \mathbf{w}) = \mathbf{u} \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{w} \]

\[ \mathbf{v} \cdot \mathbf{v} = 0 \quad \text{implies} \quad \mathbf{v} = 0 \]

Example

\[ \mathbf{u} \cdot (3\mathbf{w} - \mathbf{v}) = 3\mathbf{u} \cdot \mathbf{w} - \mathbf{u} \cdot \mathbf{v} \]
Lengths are defined naturally in terms of the dot product

\[ \mathbf{v} = (v_1, v_2) \]

\[ \ell^2 = v_1^2 + v_2^2 = \mathbf{v} \cdot \mathbf{v} \]

Notation for length of a vector is \( \| \mathbf{v} \| \). Also called the “norm” of \( \mathbf{v} \)

\[ \text{length} = \| \mathbf{v} \| = \sqrt{\mathbf{v} \cdot \mathbf{v}} \]

Vectors with length 1 are called “unit vectors”.

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Length of a vector

Example: \( \mathbf{u} = (\sqrt{3}, 4, 1) \)

Length = \( \|\mathbf{v}\| = \sqrt{\mathbf{v} \cdot \mathbf{v}} = \sqrt{(\sqrt{3})^2 + 4^2 + 1^2} = 2\sqrt{5} \)

Generalizes to higher dimensions.

\[
\|\mathbf{v}\| = \sqrt{\sum_{i=1}^{n} v_i^2}
\]

Don’t confuse “length of a vector” with the number of entries in a vector.
Law of Cosines?

\[ c^2 = a^2 + b^2 - 2ab \cos \theta \]
Law of Cosines

Generalization of the Pythagorean Theorem

\[ \|\mathbf{u} - \mathbf{v}\|^2 = \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2 - 2 \|\mathbf{u}\|\|\mathbf{v}\| \cos \theta \]

Written using the dot product, we get

\[ (\mathbf{u} - \mathbf{v}) \cdot (\mathbf{u} - \mathbf{v}) = \mathbf{u} \cdot \mathbf{u} + \mathbf{v} \cdot \mathbf{v} - 2 \|\mathbf{u}\|\|\mathbf{v}\| \cos \theta \]

\[ \mathbf{u} \cdot \mathbf{u} - 2(\mathbf{u} \cdot \mathbf{v}) + \mathbf{v} \cdot \mathbf{v} = \mathbf{u} \cdot \mathbf{u} + \mathbf{v} \cdot \mathbf{v} - 2 \|\mathbf{u}\|\|\mathbf{v}\| \cos \theta \]

\[ \mathbf{u} \cdot \mathbf{v} = \|\mathbf{u}\|\|\mathbf{v}\| \cos \theta \]
Perpendicular vectors

\[ \mathbf{u} \cdot \mathbf{v} = \| \mathbf{u} \| \| \mathbf{v} \| \cos \theta \]

The dot product of two perpendicular vectors is 0.

Example: \( \mathbf{u} = (3, -5) \) and \( \mathbf{v} = (5, 3) \) \( \rightarrow \) \( \mathbf{u} \cdot \mathbf{v} = 0 \)

The dot product is not like real multiplication because the product of two non-zero vectors can be zero.
Using the dot product

Allows us to define the angle $\theta$ between two vectors in any space dimension!

$$\cos \theta = \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\|\|\mathbf{v}\|}$$

What is the angle between $\mathbf{u} = (3, 4, -1)$ and $\mathbf{v} = (2, 0, 1)$?

$$\theta = \cos^{-1} \left( \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\|\|\mathbf{v}\|} \right) = \cos^{-1} \left( \frac{5}{\sqrt{26}\sqrt{5}} \right) = \cos^{-1} \left( \sqrt{\frac{5}{26}} \right) \approx 63^\circ$$
The equation of a line in standard form can be written as a dot product:

\[ ax + by = c \quad \rightarrow \quad (a, b) \cdot (x, y) = c \]

Points on the line are all the points \((x,y)\) whose dot product with \((a,b)\) is \(c\).