Accuracy, limiters and approximate Riemann solvers

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Riemann problem for linear systems

Solving the Riemann problem for linear problem

\[ q_t + A q_x = 0 \]

(1) Compute eigenvalues and eigenvectors of matrix \( A \)

(2) Compute characteristic variables by solving

\[ R \alpha = q_r - q_\ell \]

(3) Use eigenvalues or “speeds” to determine piecewise constant solution

\[
q(x, t) = q_\ell + \sum_{p : \lambda^p < x/t} \alpha^p r^p \\
= q_r - \sum_{p : \lambda^p > x/t} \alpha^p r^p
\]
Cell centers:

\[ x_i = a_x + (i - 1/2) \Delta x, \quad i = 1, 2, \ldots, M_x \]

Cell edges:

\[ x_{i-1/2} = a_x + (i - 1) \Delta x, \quad i = 1, 2, \ldots, M_x + 1 \]

Time step over interval \([0, T]\):

\[ t_n = n \Delta t, \quad n = 1, 2, \ldots, N_{out} \]
Update cell averages explicitly

\[ u > 0 \]

\[ Q_{i-1} \quad Q_i \quad Q_{i+1} \]
Update cell averages explicitly

In time $\Delta t$, mass in cell $C_i$ increases by shaded area:

$$\Delta x Q^n_{i+1} = \Delta x Q^n_i - u \Delta t (Q^n_i - Q^n_{i-1})$$
Update cell averages explicitly

In time $\Delta t$, mass in cell $C_i$ increases by shaded area:

$$\Delta x Q_i^{n+1} = \Delta x Q_i^n - u \Delta t \left( Q_{i+1}^n - Q_i^n \right)$$
Wave propagation viewpoint - scalar equation

\[ Q_{i}^{n+1} = Q_{i}^{n} - \frac{\Delta t}{\Delta x} \left( u^{+}(Q_{i}^{n} - Q_{i}^{n}_i) + u^{-}(Q_{i+1}^{n} - Q_{i}^{n}) \right) \]

where

\[ u^{+} = \max(u, 0), \quad u^{-} = \min(u, 0) \]

We can define waves at each interface as:

Waves: \[ \mathcal{W}_{i-1/2} \equiv Q_{i} - Q_{i-1} \]

Our scheme might look like:

\[ Q_{i}^{n+1} = Q_{i}^{n} - \frac{\Delta t}{\Delta x} \left( u^{+}\mathcal{W}_{i-1/2} + u^{-}\mathcal{W}_{i+1/2} \right) \]

“wave propagation algorithm” (R. J. LeVeque)
Wave propagation viewpoint - scalar equation

\[ Q_{i}^{n+1} = Q_{i}^{n} - \frac{\Delta t}{\Delta x} \left( u^{+}W_{i-1/2} + u^{-}W_{i+1/2} \right) \]

We can write this in terms of fluctuations:

\[ A^{+} \Delta Q_{i-1/2} = u^{+}W_{i-1/2} \]
\[ A^{-} \Delta Q_{i+1/2} = u^{-}W_{i+1/2} \]

\[ Q_{i}^{n+1} = Q_{i}^{n} - \frac{\Delta t}{\Delta x} \left( A^{+} \Delta Q_{i-1/2} + A^{-} \Delta Q_{i+1/2} \right) \]

The first order term in the update used by Clawpack and GeoClaw
Wave propagation viewpoint - systems

\[ Q_i^{n+1} = Q_i^n - \frac{\Delta t}{\Delta x} \left( \sum_{p=1}^{m} (\lambda^p)^+ W_{i-1/2}^p + \sum_{p=1}^{m} (\lambda^p)^- W_{i+1/2}^p \right) \]

where the waves are now defined from an eigenvalue decomposition of the jump in value at each interface

Waves: \[ W_{i-1/2}^p \equiv \alpha^p r^p \]

Written in terms of fluctuations:

\[ A^+ \Delta Q_{i-1/2} \equiv \sum_{p=1}^{m} (\lambda^p)^+ W_{i-1/2}^p \quad (\lambda^p)^+ = \max(\lambda^p, 0) \]

\[ A^- \Delta Q_{i-1/2} \equiv \sum_{p=1}^{m} (\lambda^p)^- W_{i-1/2}^p \quad (\lambda^p)^- = \min(\lambda^p, 0) \]
Riemann problem for scalar advection - wave propagation approach

do  i = 2-mbc,mx+mbc
   delta = ql(i,1) - qr(i-1,1)

   # Speeds
   s(i,1) = ubar

   # Waves
   wave(i,1,1) = delta

   # Fluctuations
   amdq(i,1) = min(s(i,1), 0.d0) * wave(i,1,1)
   apdq(i,1) = max(s(i,1), 0.d0) * wave(i,1,1)
endo
Given an exact solution, we can also construct waves, speeds and fluctuations.
Numerical fluxes can be written in terms of fluctuations:

\[ F_{i-1/2} \equiv f(Q_i) - A^+ \Delta Q_{i-1/2} \]

or

\[ F_{i-1/2} \equiv f(Q_{i-1}) + A^- \Delta Q_{i-1/2} \]

Flux differences can be expressed in terms of left going and right going fluctuations:

\[ F_{i+1/2} - F_{i-1/2} = A^+ \Delta Q_{i-1/2} + A^- \Delta Q_{i+1/2} \]
Roe linearization for the non-linear case

\[ qt + f(q)_x = 0 \]

We solve a linearized system at each cell interface, at each time step

\[ qt + f'(\hat{q})q_x = 0 \quad \text{and} \quad qt + A(\hat{q})q_x = 0 \]

for “Roe averaged” values \( \hat{q} \).

For conservation, we need \( \hat{q} \) to satisfy:

\[ f(q_r) - f(q_e) = f'(\hat{q})(q_r - q_e) \]

P. Roe (JCP, 1981) showed an approach for many important systems.
Roe averaged values for SWE

• Compute Roe averaged values:

\[ \hat{h} = \frac{h_\ell + h_r}{2}, \quad \hat{u} = \frac{\sqrt{h_\ell}u_\ell + \sqrt{h_r}u_r}{\sqrt{h_\ell} + \sqrt{h_r}} \]

• Evaluate eigenvalues and eigenvectors at these values.

• Compute waves, speeds and fluctuations as in the linear case. Does not require the nonlinear root-finder.

Roe averages are also available for the Euler equations and other important physical systems.

Other approximate Riemann solvers are available.
do i = 2-mbc,mx+mbc

\begin{verbatim}
# compute Roe-averaged quantities:
u_roe = (ur/sqrt(hr) + ul/sqrt(hl))/(sqrt(hl) + sqrt(hr))
h_mean = (hl + hr)/2.d0
sqrtgh_roe = sqrt(grav*h_mean)

# wave speeds
s(i,1) = u_roe - sqrtgh_roe
s(i,2) = u_roe + sqrtgh_roe

# compute coeffs in the evector expansion of delta(1),delta(2)
a1 =(-delta(2) + (u_roe + sqrtgh_roe)*delta(1))/(2*sqrtgh_roe)
a2 = (delta(2) - (u_roe - sqrtgh_roe)*delta(1))/(2*sqrtgh_roe)

# finally, compute the waves.
wave(i,1,1) = a1
wave(i,2,1) = a1*(u_roe - sqrtgh_roe)
wave(i,1,2) = a2
wave(i,2,2) = a2*(u_roe + sqrtgh_roe)
\end{verbatim}

\begin{verbatim}
do mw=1,mwaves
  amdq(i,m) = amdq(i,m) + min(s(i,mw), 0.d0) * wave(i,m,mw)
  apdq(i,m) = apdq(i,m) + max(s(i,mw), 0.d0) * wave(i,m,mw)
endo
\end{verbatim}
Question: There is only one shock and one rarefaction, but we solve a Riemann problem (either exactly, or approximately) at each cell interface. What happens in the smooth regions?

Answer: In smooth regions, shocks/rarefactions are weak. They only have strength on the order of the mesh cell size, i.e.

\[ q_r - q_\ell \sim O(\Delta x) \]
Upwind method

First order scheme (200 points)
Upwind method

First order scheme (200 points)
The upwind method

\[ Q_i^{n+1} = Q_i^n - \frac{\Delta t}{\Delta x} \left( A^+ \Delta Q_{i-1/2} + A^- \Delta Q_{i+1/2} \right) \]

\[ = Q_i^n - \frac{\Delta t}{\Delta x} \bar{u}(Q_i^n - Q_{i-1}), \quad \text{for} \quad \bar{u} > 0 \]

is only a first order approximation, but gives a good second order approximation to the equation

\[ q_t + \bar{u}q_x = \frac{\bar{u} \Delta x}{2} \left( 1 - \frac{\bar{u} \Delta t}{\Delta x} \right) q_{xx} \]

The “diffusion term” is proportional to the mesh spacing and the Courant number.
Why not include these “diffusion” terms in the numerical scheme to get better accuracy?

\[ q_t + u q_x = \frac{u \Delta x}{2} \left( 1 - \frac{u \Delta t}{\Delta x} \right) q_{xx} \quad u > 0 \]

Re-arranging terms, we get the second order “Lax-Wendroff” method.
The Lax-Wendroff method

\[ Q_{i+1}^n = Q_i^n - \frac{u \Delta t}{\Delta x} (Q_i^n - Q_{i-1}^n) - \frac{1}{2} \frac{u \Delta t}{\Delta x} \left( 1 - \frac{u \Delta t}{\Delta x} \right) \left( (Q_{i+1}^n - Q_i^n) - (Q_i^n - Q_{i-1}^n) \right) \]

- **Upwind term**
- **Second order correction**

The Lax-Wendroff method gives a third order approximation to the modified equation

\[ q_t + u q_x = -\frac{u (\Delta x)^2}{6} \left( 1 - \left( \frac{u \Delta t}{\Delta x} \right)^2 \right) q_{xxx} \]

*Errors are dispersive*
Lax Wendroff method

\[
Q_{i}^{n+1} = Q_{i}^{n} - \frac{u \Delta t}{\Delta x} (Q_{i}^{n} - Q_{i-1}^{n}) - \frac{1}{2} \frac{u \Delta t}{\Delta x} \left(1 - \frac{u \Delta t}{\Delta x}\right) ((Q_{i+1}^{n} - Q_{i}^{n}) - (Q_{i}^{n} - Q_{i-1}^{n}))
\]

\[
Q_{i}^{n+1} = Q_{i}^{n} - \frac{u \Delta t}{\Delta x} (Q_{i}^{n} - Q_{i-1}^{n}) - \frac{1}{2} \frac{u \Delta t}{\Delta x} (\Delta x - u \Delta t) (\sigma_{i}^{n} - \sigma_{i-1}^{n})
\]

\[
\sigma_{i}^{n} = \frac{Q_{i+1}^{n} - Q_{i}^{n}}{\Delta x}
\]

\[
\mathcal{W}_{i-1/2}^{n} = Q_{i}^{n} - Q_{i-1}^{n}
\]
For $u > 0$:

$$Q_{i}^{n+1} = Q_{i}^{n} - \frac{\Delta t}{\Delta x} (u \omega_{i-1/2}^{n}) - \frac{1}{2} \frac{u \Delta t}{\Delta x} \left(1 - \frac{u \Delta t}{\Delta x}\right) \left(\omega_{i+1/2}^{n} - \omega_{i-1/2}^{n}\right)$$

In wave propagation form, we can write this as:

$$Q_{i}^{n+1} = Q_{i}^{n} - \frac{\Delta t}{\Delta x} \left(\frac{A^+ \Delta Q_{i-1/2}^{n} + A^- \Delta Q_{i+1/2}^{n}}{2}\right) - \frac{\Delta t}{\Delta x} \left(\mathcal{F}_{i+1/2}^{n} - \mathcal{F}_{i-1/2}^{n}\right)$$

where second order correction terms are defined as

$$\mathcal{F}_{i-1/2}^{n} \equiv \frac{1}{2} |u| \left(1 - \frac{|u| \Delta t}{\Delta x}\right) \omega_{i-1/2}^{n}$$
Wave propagation algorithm for systems

For systems (both linear and nonlinear), we have the update formula

\[ Q_{i}^{n+1} = Q_{i}^{n} - \frac{\Delta t}{\Delta x} (A^+ \Delta Q_{i-1/2} + A^- \Delta Q_{i+1/2}) - \frac{\Delta t}{\Delta x} (F_{i+1/2} - F_{i-1/2}) \]

where the second order corrections are defined as

\[ F_{i-1/2} = \frac{1}{2} \sum_{p=1}^{m} |\lambda^p| \left( 1 - \frac{\Delta t}{\Delta x} |\lambda^p| \right) W_{i-1/2}^p \]
The Lax-Wendroff method

Second order terms included (200 points)
The Lax-Wendroff method

Second order terms included (200 points)
First order REA algorithm

Cell averages and piecewise constant reconstruction:

After evolution:
Second order REA algorithm

Cell averages and piecewise linear reconstruction:

After evolution:
Oscillations

Any of these slope choices will give oscillations near discontinuities.

Ex: Lax-Wendroff:

Average value increases

Lax-Wendroff slope
High resolution methods

Want to use slope where solution is smooth for “second-order” accuracy.

Where solution is not smooth, adding slope corrections gives oscillations.

Limit the slope based on the behavior of the solution.

$$\sigma_i^n = \left(\frac{Q_{i+1}^n - Q_i^n}{\Delta x}\right) \Phi_i^n.$$  

$$\Phi = 1 \implies \text{Lax-Wendroff},$$  

$$\Phi = 0 \implies \text{upwind}.$$
Minmod slope

\[
\text{minmod}(a, b) = \begin{cases} 
  a & \text{if } |a| < |b| \text{ and } ab > 0 \\
  b & \text{if } |b| < |a| \text{ and } ab > 0 \\
  0 & \text{if } ab \leq 0
\end{cases}
\]

Slope:

\[
\sigma_i^n = \text{minmod}\left(\frac{(Q_i^n - Q_{i-1}^n)}{\Delta x}, \frac{(Q_{i+1}^n - Q_i^n)}{\Delta x}\right)
\]

\[
= \left(\frac{Q_{i+1}^n - Q_i^n}{\Delta x}\right) \Phi(\theta_i^n)
\]

where

\[
\theta_i^n = \frac{Q_i^n - Q_{i-1}^n}{Q_{i+1}^n - Q_i^n}
\]

\[
\Phi(\theta) = \text{minmod}(\theta, 1) \quad 0 \leq \Phi \leq 1
\]
Minmod reconstruction

Lax-Wendroff reconstruction:

Minmod reconstruction:
Limiters

Linear methods:

- upwind: \( \phi(\theta) = 0 \)
- Lax-Wendroff: \( \phi(\theta) = 1 \)
- Beam-Warming: \( \phi(\theta) = \theta \)
- Fromm: \( \phi(\theta) = \frac{1}{2} (1 + \theta) \)

High-resolution limiters:

- minmod: \( \phi(\theta) = \minmod(1, \theta) \)
- superbee: \( \phi(\theta) = \max(0, \min(1, 2\theta), \min(2, \theta)) \)
- MC: \( \phi(\theta) = \max(0, \min((1 + \theta)/2, 2, 2\theta)) \)
- van Leer: \( \phi(\theta) = \frac{\theta + |\theta|}{1 + |\theta|} \)
Hierarchy of methods for systems

Waves
\[ \mathcal{W}_{i-1/2}^p \equiv \alpha^p r^p \]

Fluctuations
\[ \mathcal{A}^+ \Delta Q_{i-1/2} \equiv \sum_{p=1}^m (\lambda^p)^+ \mathcal{W}_{i-1/2} \]
\[ \mathcal{A}^- \Delta Q_{i-1/2} \equiv \sum_{p=1}^m (\lambda^p)^- \mathcal{W}_{i-1/2} \]

Upwind
\[ Q_{i}^{n+1} = Q_{i}^n - \frac{\Delta t}{\Delta x} \left( \mathcal{A}^+ \Delta Q_{i-1/2} + \mathcal{A}^- \Delta Q_{i+1/2} \right) \]

Lax Wendroff
\[ Q_{i}^{n+1} = Q_{i}^n - (\text{upwind}) - \frac{\Delta t}{\Delta x} (\mathcal{F}_{i+1/2} - \mathcal{F}_{i-1/2}) \]

High Resolution
\[ Q_{i}^{n+1} = Q_{i}^n - (\text{upwind}) - \frac{\Delta t}{\Delta x} \left( \tilde{\mathcal{F}}_{i+1/2} - \tilde{\mathcal{F}}_{i-1/2} \right) \]

Fluctuations
\[ \mathcal{F}_{i-1/2} \equiv \frac{1}{2} \sum_{p=1}^m |\lambda^p| \left( 1 - \frac{\Delta t}{\Delta x} |\lambda^p| \right) \mathcal{W}_{i-1/2}^p \]
\[ \tilde{\mathcal{F}}_{i-1/2} \equiv \frac{1}{2} \sum_{p=1}^m |\lambda^p| \left( 1 - \frac{\Delta t}{\Delta x} |\lambda^p| \right) \Phi \mathcal{W}_{i-1/2}^p \]
High resolution methods

- Wave limiters reduce oscillations near discontinuities, but preserve second order accuracy in smooth regions.
- Methods are no longer formally second order accurate, but are “high resolution”.
- Often magnitude of the error is reduced by the use of limiters.
- Useful even for linear problems such as advection.
- Clawpack has several limiters available.
- Limiters only affect second order correction terms; first order method does not use limiters.
High resolution methods

Second order method with limiter (200 points)
High resolution methods

Second order method with limiter (200 points)