Introduction to Finite Volume methods for Hyperbolic problems

Donna Calhoun

Boise State University
The shallow water wave equations, given by

\[ h_t + (uh)_x = 0 \]

\[ (uh)_t + (hu^2 + \frac{1}{2}gh^2)_x = 0 \]

is an example of a system of equations written in conservative form. More generally, we can write PDEs in conservative form as

\[ q_t + f(q)_x = 0 \]

These are typically derived from conservation laws for mass, momentum, energy, species, and so on.
• Based on solving the conservative form of the shallow water wave equations using a finite volume method.

• At the heart of many finite volume methods is a Riemann solver which is used to compute numerical fluxes.

• In GeoClaw, these are stored in files like `rpn2_geo.f` and `rpt2_geo.f`
Finite volume method

Assume a conservation law of the form

\[ q_t + f(q)_x = 0 \]

Define cell averages over the interval \( C_i = [x_{i-1/2}, x_{i+1/2}] \)

\[ Q_i^n = \frac{1}{\Delta x} \int_{C_i} q(x, t_n) \, dx \]

How does the average evolve?

\[ \frac{d}{dt} \int_{C_i} q(x, t) \, dx = - \int_{C_i} \frac{d}{dx} f(q(x, t)) \, dx \]

\[ = f(q(x_{i-1/2}, t)) - f(q(x_{i+1/2}, t)) \]
Finite volume method

Evolution of the cell average value:

$$\frac{d}{dt} \int_{C_i} q(x, t) \, dx = f(q(x_{i-1/2}, t)) - f(q(x_{i+1/2}, t))$$

Integrate in time

$$\int_{C_i} q(x, t_{n+1}) \, dx = \int_{C_i} q(x, t_n) \, dx$$

$$+ \int_{t_n}^{t_{n+1}} [f(q(x_{i-1/2}, t)) - f(q(x_{i+1/2}, t))] \, dt$$
Using numerical fluxes, we use the update formula:

\[
Q_{i}^{n+1} = Q_{i}^{n} - \frac{\Delta t}{\Delta x} \left[ F_{i+1/2}^{n} - F_{i-1/2}^{n} \right]
\]

Written as

\[
\frac{Q_{i}^{n+1} - Q_{i}^{n}}{\Delta t} + \frac{F_{i+1/2}^{n} - F_{i-1/2}^{n}}{\Delta x} = 0
\]

this form resembles the conservation law:

\[
q_{t} + f(q)_{x} = 0
\]
Numerical fluxes

We want to approximate the numerical flux.

\[ F_{i-1/2}^n \approx \frac{1}{\Delta t} \int_{t_n}^{t_{n+1}} f(q(x_{i-1/2}, t)) \, dt \]

For an explicit time stepping scheme, we try to find formulas for the flux of the form

\[ F_{i-1/2}^n = \mathcal{F}(Q_i^n, Q_{i-1}^n) \]
1d Riemann problem

At each cell interface, solve the hyperbolic problem with special initial data, i.e.

\[ q_t + f(q)_x = 0 \]

subject to

\[ q(x, 0) = \begin{cases} Q_{i-1} & x < x_{i-1/2} \\ Q_i & x > x_{i-1/2} \end{cases} \]
Solve the conservation law with piecewise constant initial data
1d Riemann problem

Numerical flux at cell interface is then approximated by

\[ F_{i-1/2} = f(q^*) \]

This is the classical Godunov approach for solving hyperbolic conservation laws.

- Resolves shocks and rarefactions
Integrating over entire domain, we have

\[ \frac{d}{dt} \int_{x_a}^{x_b} q(x, t) \, dx = - \int_{x_a}^{x_b} (f(q))_x \, dx = f(q(x_a, t)) - f(q(x_b, t)) \]

**Discrete case**

\[
\sum_{i=1}^{M} Q_{i}^{n+1} = \sum_{i=1}^{M} Q_{i}^{n} - \frac{\Delta t}{\Delta x} \sum_{i=1}^{M} (F_{i+1/2} - F_{i-1/2})
\]

\[ = \sum_{i=1}^{M} Q_{i}^{n} - \frac{\Delta t}{\Delta x} (F_{M+1/2} - F_{1/2}) \]

**Quantities are conserved up to fluxes at domain boundaries.**
Scalar advection

Consider the constant initial value problem

\[ q_t + \bar{u} q_x = 0 \]

\[ q(x, 0) = \eta(x) \]

It is easy to verify that

\[ q(x, t) = \eta(x - \bar{u}t) \]

solves the initial value problem.
Scalar advection

We can describe the problem in terms of how the solution behaves along curves in the x-t plane.

We might look for curves \( \sigma = (X(t), t) \) along which the solution is constant or

\[
\frac{d}{dt} q(X(t), t) = 0
\]

Then we would get

\[
\frac{d}{dt} q(X(t), t) = q_x(X(t), t)X'(t) + q_t(X(t), t) = 0
\]
Characteristic curves

\[
\frac{d}{dt} q(X(t), t) = q_x(X(t), t) X'(t) + q_t(X(t), t) = 0
\]

But this is true only if

\[
X'(t) = \bar{u}
\]

or

\[
X(t) = \bar{u}t + X_0
\]

Solution is constant along characteristic curves. For \( \bar{u} > 0 \),

\[
(t > 0, \bar{u}t + X_0, t)
\]
The solution can be traced back along characteristics. That is, $q(x, t)$ can be found by determining the $X_0$ from which the solution propagated. Solve

$$x = \bar{u}t + X_0 \quad \rightarrow \quad X_0 = x - \bar{u}t$$

or

$$q(x, t) = q(X_0, 0) = q(x - \bar{u}t, 0)$$
Consider the scalar advection equation:

\[ q_t + \bar{u} q_x = 0 \]

The solution travels along characteristic rays in the \((x,t)\) plane given by \((x - X_0)/t = \bar{u}\). For \(u < 0\):

\[ t = 1 \rightarrow \]

\[ t = 0 \rightarrow \]

**Scalar advection**
Riemann problem for scalar advection

\[ q_t + \bar{u}q_x = 0 \]

subject to particular initial conditions

\[ q(x, 0) = \begin{cases} q_\ell & x < 0 \\ q_r & x > 0 \end{cases} \]

Solution:

\[ q(x, t) = \begin{cases} q_\ell & x - \bar{u}t < 0 \\ q_r & x - \bar{u}t > 0 \end{cases} \]
Scalar Riemann Problem

\[ q_t + \bar{u}q_x = 0 \]

subject to initial conditions

\[ q(x, 0) = \begin{cases} 
q_L & x < 0 \\
q_R & x > 0 
\end{cases} \]

Solution:

\[ q(x, t) = \begin{cases} 
q_L & x/t < \bar{u} \\
q_R & x/t > \bar{u} 
\end{cases} \]
Scalar constant coefficient advection

Discontinuity propagates at speed $\bar{u}$ and has strength $q_r - q_\ell$
Solving constant coefficient linear systems

\[ q_t + A q_x = 0, \quad A \in \mathbb{R}^{m \times m} \]

We assume that \( A \) has a complete set of eigenvectors and real eigenvalues and so can be written as

\[ A = R \Lambda R^{-1} \]

\[ R = [r^1, r^2, \ldots r^m] \quad \Lambda = \text{diag}(\lambda^1, \lambda^2, \ldots \lambda^m) \]

Examples: Linearized shallow water wave equations, Euler equations, two way wave-equation, ...
Solving a constant coefficient system

\[ q_t + A q_x = 0 \quad \rightarrow \quad q_t + R \Lambda R^{-1} q_x = 0, \quad A \in \mathbb{R}^{m \times m} \]

Define characteristic variables \( \omega \in \mathbb{R}^m \) as

\[ \omega(x, t) = R^{-1} q(x, t), \quad \omega(x, 0) = R^{-1} q(x, 0) \]

Characteristic equations decouple into \( m \) scalar equations:

\[ \omega_t^p + \lambda^p \omega_x^p = 0, \quad p = 1, 2, \ldots, m \]

Solution to characteristic equations are given by

\[ \omega^p(x, t) = \omega^p(x - \lambda^p t, 0) \]
Solving a constant coefficient systems

\[ q_t + A q_x = 0 \quad \rightarrow \quad \omega_t^p + \lambda^p \omega_x^p = 0 \]

Solution for general initial conditions \( q(x, 0) : \)

\[ q(x, t) = R \omega(x, t) = \sum_{p=1}^{m} \omega^p(x, t) r^p \]

\[ = \sum_{p=1}^{m} \omega^p(x - \lambda^p t, 0) r^p \]

\[ \ell^p q(x - \lambda^p t, 0) = \omega^p(x - \lambda^p t, 0) \]

Donna Calhoun, BSU (2016)
Riemann problem for systems

Assume a constant coefficient system:

\[ q_t + A q_x = 0, \quad q \in \mathbb{R}^3 \]

with piecewise constant initial data:

\[ q(x, 0) = \begin{cases} 
q_\ell & x < 0 \\
q_r & x > 0 
\end{cases} \]

which can be decomposed as:

\[ q_\ell = \sum_{p=1}^{3} \omega_\ell^p r^p \quad q_r = \sum_{p=1}^{3} \omega_r^p r^p \]
Riemann problem for systems

\[ q_t + A q_x = 0, \quad q \in \mathbb{R}^3 \]

\[ t > 0 \]

\[ t = 0 \]

\[ q_\ell, q_r \]

\[ x = 0 \]

Piecewise constant initial data

\[ q(x_1, t) = \omega^1_\ell r^1 + \omega^2_\ell r^2 + \omega^3_\ell r^3 \]

\[ q_\ell = \sum_{p=1}^{m} \omega^p_\ell r^p \]

\[ q_r = \sum_{p=1}^{m} \omega^p_r r^p \]
Riemann problem for systems

\[ q_t + A q_x = 0 \]

\[ q(x_2, t) = \omega_r^1 r^1 + \omega_\ell^2 r^2 + \omega_\ell^3 r^3 \]

\[ q_\ell = \sum_{p=1}^{m} \omega_\ell^p r^p \quad q_r = \sum_{p=1}^{m} \omega_r^p r^p \]
Riemann problem for systems

\[ q_t + A q_x = 0 \]

- \[ q(x_3, t) = \omega^1 r^1 + \omega^2 r^2 + \omega^3 r^3 \]

\[ q_\ell = \sum_{p=1}^{m} \omega^p_\ell r^p \]

\[ q_r = \sum_{p=1}^{m} \omega^p_r r^p \]
Riemann problem for systems

\[ qt + A \, q_x = 0 \]

\[ q(x_4, t) = \omega_r^1 \, r^1 + \omega_r^2 \, r^2 + \omega_r^3 \, r^3 \]

\[ q_\ell = \sum_{p=1}^{m} \omega_\ell^p \, r^p \]

\[ q_r = \sum_{p=1}^{m} \omega_r^p \, r^p \]
Riemann problem for systems

\[ q_t + A q_x = 0 \]

\[ t > 0 \]

\[ t = 0 \]

\[ q(x_1, t) = \omega^1_r r^1 + \omega^2_r r^2 + \omega^3_r r^3 \]

\[ q(x_2, t) = \omega^1_l r^1 + \omega^2_l r^2 + \omega^3_l r^3 \]

\[ q(x_3, t) = \omega^1_r r^1 + \omega^2_r r^2 + \omega^3_r r^3 \]

\[ q(x_4, t) = \omega^1_r r^1 + \omega^2_r r^2 + \omega^3_r r^3 \]
Riemann problem for systems

\[ q_t + A q_x = 0 \]

\[ t > 0 \]

\[ t = 0 \]

\[ q^p(x, t) \]

\[ q_\ell \quad q_r \]
Riemann problem for systems

\[ q_t + A q_x = 0 \]

\[ q^p(x, t) \]

\[ q(x_1, t) = \omega^1_r r^1 + \omega^2_r r^2 + \omega^3_r r^3 \]

\[ q(x_2, t) = \omega^1_r r^1 + \omega^2_r r^2 + \omega^3_r r^3 \]

\[ q(x_2, t) - q(x_1, t) = (\omega^1_r - \omega^1_l) r^1 \]
Riemann problem for systems

\[ q_t + A q_x = 0 \]

\[ q^p(x, t) \]

\[ q(x_2, t) = \omega^1_r r^1 + \omega^2_\ell r^2 + \omega^3_\ell r^3 \]

\[ q(x_3, t) = \omega^1_r r^1 + \omega^2_r r^2 + \omega^3_\ell r^3 \]

\[ q(x_3, t) - q(x_2, t) = (\omega^2_r - \omega^2_\ell) r^2 \]
Riemann problem for systems

\[ q_t + A q_x = 0 \]

\[ q^p(x, t) \]

\[ q(x_3, t) = \omega_1^1 r^1 + \omega_2^2 r^2 + \omega_3^3 r^3 \]

\[ q(x_4, t) = \omega_1^1 r^1 + \omega_2^2 r^2 + \omega_3^3 r^3 \]

\[ q(x_4, t) - q(x_3, t) = (\omega_3^3 - \omega_3^3) r^3 \]
Riemann problem for systems

\[ q_t + A q_x = 0 \]

\[ q^p(x, t) \]

\[ q(x_4, t) - q(x_1, t) = \sum_{p=1}^{3} (\omega^p_r - \omega^p_\ell) r^p \equiv \sum_{p=1}^{3} \alpha^p r^p \]

\[ R \alpha = q_r - q_\ell \]
Riemann problem for systems

Solving the Riemann problem for linear problem

$$q_t + A q_x = 0$$

(1) Compute eigenvalues and eigenvectors of matrix $A$

(2) Compute characteristic variables by solving

$$R \alpha = q_r - q_{\ell}$$

(3) Use eigenvalues or “speeds” to determine piecewise constant solution

$$q(x, t) = q_{\ell} + \sum_{p : \lambda^p < x/t} \alpha^p r^p$$

$$= q_r - \sum_{p : \lambda^p > x/t} \alpha^p r^p$$
Numerical solution

\[ qt + f(q)x = 0, \quad f(q) = Aq \]

\[ Q_{i}^{n+1} = Q_{i}^{n} - \frac{\Delta t}{\Delta x} \left[ F_{i+1/2}^{n} - F_{i-1/2}^{n} \right] \]

Decompose jump in Q at the interface into waves:

\[ q^* = Q_{i-1} + \alpha^1 r^1 = Q_{i} - \alpha^3 r^3 - \alpha^2 r^2 \]

\[ F_{i-1/2} \approx \frac{1}{\Delta t} \int_{t}^{t+\Delta t} f(q(x_{i-1/2}, t)) dt = Aq^* \]
Example: Linearized shallow water

\[ q_t + A q_x = 0, \quad A = \begin{pmatrix} U & H \\ g & U \end{pmatrix}, \quad q = \begin{pmatrix} h \\ u \end{pmatrix} \]

Characteristic information:

Eigenvalues: \[ \lambda^1 = U - \sqrt{gH}, \quad \lambda^2 = U + \sqrt{gH} \]

Eigenvectors: \[ r^1 = \begin{pmatrix} -\sqrt{gH} \\ g \end{pmatrix}, \quad r^2 = \begin{pmatrix} \sqrt{gH} \\ g \end{pmatrix} \]
Example: Linearized shallow water

Characteristic variables: \( R\alpha = q_r - q_\ell \)

Define: \( \delta = q_r - q_\ell \) \( \rightarrow \) \( \delta^1 = h_r - h_\ell \)
\( \delta^2 = u_r - u_\ell \)

\[
\begin{pmatrix}
\alpha_1 \\
\alpha_2
\end{pmatrix} = \frac{1}{2gH} \begin{pmatrix}
-\sqrt{gH} & H \\
\sqrt{gH} & H
\end{pmatrix} \begin{pmatrix}
\delta_1 \\
\delta_2
\end{pmatrix}
\]

\[
q(x, t) = \begin{cases} 
q_\ell = \begin{pmatrix} h_\ell \\
u_\ell \end{pmatrix} & x/t < U - \sqrt{gH} \\
q_\ell + \alpha^1 r^1 & U - \sqrt{gH} < x/t < U + \sqrt{gH} \\
q_r = \begin{pmatrix} h_r \\
u_\ell \end{pmatrix} & x/t > U + \sqrt{gH}
\end{cases}
\]
Linear shallow water wave equations

Initial height and velocity

\[ W_2 \]

\[ W_1 \]

\[ W^1 \] \[ W^2 \]
Extending to nonlinear systems

\[ q^* - q_\ell = \alpha^1 r^1 \]

\[ A(q^* - q_\ell) = \alpha^1 A r^1 \]

\[ A(q^* - q_\ell) = \lambda^1 (\alpha^1 r^1) \]

\[ A(q^* - q_\ell) = \lambda^1 (q^* - q_\ell) \]

\[ f(q) = Aq \quad \rightarrow \quad f(q^*) - f(q_\ell) = \lambda^1 (q^* - q_\ell) \]

Rankine-Hugoniot condition for the constant coefficient linear system
Rankine-Hugoniot conditions

For a 2x2 linear system, we have

\[ A(q^* - q_\ell) = \lambda^1(q^* - q_\ell) \]
\[ A(q_r - q^*) = \lambda^2(q_r - q^*) \]

For \( f(q) = Aq \), we can write this as:

\[ f(q^*) - f(q_\ell) = \lambda^1(q^* - q_\ell) \]
\[ f(q_r) - f(q^*) = \lambda^2(q_r - q^*) \]

The left and right states \( q_\ell \) and \( q_r \) as “connected” by an intermediate state \( q^* \).
We could have asked “Find an intermediate state \( q^* \) such that

\[
\begin{align*}
    f(q^*) - f(q_\ell) &= \lambda^1 (q^* - q_\ell) \\
    f(q_r) - f(q^*) &= \lambda^2 (q_r - q^*)
\end{align*}
\]

For \( f(q) = Aq \), this leads to the eigenvalue problem that we solved.
Extending to nonlinear systems

Reminder: Solutions to the constant coefficient linear system travel along characteristic curves $(X(t), t)$:

$$\frac{d}{dt} q(X(t), t) = q_t + X'(t)q_x = 0$$

$$X'(t)q_x = Aq_x$$

$X'(t)$ must be an eigenvalue of $A$, i.e. $X'(t) = \lambda^1, \lambda^2$

Solution remains constant along straight lines
Nonlinear shallow water wave equations

\[ q_t + f(q)_x = 0 \]

where

\[ q = \begin{pmatrix} h \\ hu \end{pmatrix}, \quad f(q) = \begin{pmatrix} hu \\ hu^2 + \frac{1}{2}gh^2 \end{pmatrix} \]

for smooth solutions, this can also be written as

\[ q_t + f'(q)q_x = 0 \]

where

\[ f'(q) = \begin{pmatrix} 0 & 1 \\ -u^2 + gh & 2u \end{pmatrix} \]

is the flux Jacobian matrix.
What changes in the nonlinear case?

\[ q_t + f(q)_x = 0 \]

We can still ask “Are there characteristic curves on which the solution remains constant?

\[ \frac{d}{dt} q(X(t), t) = q_t + X'(t)q_x = 0 \]

For smooth solutions, we have

\[ q_t + f'(q)q_x = 0 \]

where \( f'(q) \in \mathcal{R}^{m \times m} \) is the flux Jacobian matrix.

\[ f'(q)q_x = X'(t)q_x \]

*Characteristics are governed by eigenvalues of the flux Jacobian*
Shallow water wave equations

\[ q = \begin{pmatrix} h \\ hu \end{pmatrix}, \quad f(q) = \begin{pmatrix} hu \\ hu^2 + \frac{1}{2} gh^2 \end{pmatrix} \]

\[ f'(q) = \begin{pmatrix} 0 & 1 \\ -u^2 + gh & 2u \end{pmatrix} \]

Eigenvalues and eigenvectors of the flux Jacobian \( f'(q) \):

\[ \lambda^1 = u - \sqrt{gh}, \quad \lambda^2 = u + \sqrt{gh} \]

\[ r^1 = \begin{pmatrix} 1 \\ u - \sqrt{gh} \end{pmatrix}, \quad r^2 = \begin{pmatrix} 1 \\ u + \sqrt{gh} \end{pmatrix} \]

*Eigenvalues and eigenvectors depend on \( q \)!*
What can happen?

\[
\frac{x}{t} = u_l - \sqrt{g h_l}
\]

\[
\frac{x}{t} = u_l + \sqrt{g h_l}
\]

1-characteristics

\[
\frac{x}{t} = u_r - \sqrt{g h_r}
\]

\[
\frac{x}{t} = u_r + \sqrt{g h_r}
\]

2-characteristics
Consider the case where $\lambda^p(q^L) > \lambda^p(q^R)$:

Using the conservation law, we can write

$$\frac{d}{dt} \int_{x_1}^{x_1+\Delta x} q(x, t) \, dx + \int_{x_1}^{x_1+\Delta x} f(q(x, t))_x \, dx = 0$$

$$\frac{q^L - q^R}{\Delta t} + \Delta x (f(q^R) - f(q^L)) = 0$$

Assume that left and right states are constant in this infinitesimal box.
Riemann problem for SWE

This leads to

\[ f(q_r) - f(q_\ell) = \frac{\Delta x}{\Delta t} (q_r - q_\ell) \]

This is the required jump condition across shocks. More generally we can write this condition as

\[ f(q_r) - f(q_\ell) = s(q_r - q_\ell) \]

where \( s \) is the shock speed.
Consider the case where $\lambda^p(q_l) < \lambda^p(q_r)$:

Let $\xi = \frac{x}{t}$ be the slope of the characteristic. We need to find $q(\xi)$ for $\lambda^1(q_l) < \xi < \lambda^1(q_r)$. Recall that $f'(q(\xi))q'(\xi) = \xi q'(\xi)$. Then

\[
\xi = \lambda^1(q(\xi))
\]

\[
\rightarrow 1 = \nabla \lambda^1(q(\xi)) \cdot q'(\xi)
\]

\[
= \alpha(\xi) \nabla \lambda^1(q(\xi)) \cdot r^1(q(\xi))
\]

\[
\rightarrow q'(\xi) = \frac{r^1(q(\xi))}{\nabla \lambda^1(q(\xi)) \cdot r^1(q(\xi))}
\]

The denominator is never 0!

Solve resulting ODE to get $q(\xi)$ in the centered rarefaction.
Solve the system of two ODEs (for SWE):

\[
q'(\xi) = \frac{r^1(q(\xi))}{\nabla \lambda^1(q(\xi)) \cdot r^1(q(\xi))}
\]

subject to

\[
\begin{align*}
\xi_1 &= \lambda^1(q_\ell), & q(\xi_1) &= q_\ell \\
\xi_2 &= \lambda^1(q_r), & q(\xi_2) &= q_r
\end{align*}
\]

Use Riemann invariants to solve for unknown constants.
Find a state $q^*$ such that $q_\ell$ is connected to $q^*$ by a physically correct 1-shock wave or 1-rarefaction, and $q_r$ is connected to $q^*$ by a physically correct 2-shock or 2-rarefaction.

The need to find a state $q^*$ that simultaneously satisfies both conditions above means we have to solve something...
Curves represent states that can be connected to \( q_r \) or \( q_\ell \) by a shock or a rarefaction. Use a nonlinear root-finder to find the middle state \( q^* \).

- Determine the structure of the rarefaction (if there is one).
The structure of the Riemann solution depends on the initial conditions.
Can we avoid the nonlinear solve?
How does this extend to the two dimensional shallow water equations?
How does GeoClaw use Riemann solvers?

For details, see *Finite Volume Methods for Hyperbolic Problems*, R. J. LeVeque (*Cambridge University Press, 2002*).