Math 465/565, Fall 2013

Due Thursday December 5th, start of class

For each problem, please be as thoughtful as you can. Describe clearly what you are demonstrating, and when creating a plot, be sure to include titles, axis labels, a legend (where appropriate).

Math 465 students may do Math 565 only problems for extra credit.

1. (Lagrange Polynomials) Let \( x_0 = -3, \ x_1 = 0, \ x_2 = e, \) and \( x_3 = \pi. \)
   (a) Determine the formulas for the Lagrange polynomials \( \ell_0(x), \ell_1(x), \ell_2(x) \) and \( \ell_3(x) \) associated with the given interpolating points.
   (b) Plot \( \ell_0(x), \ell_1(x), \ell_2(x) \) and \( \ell_3(x) \) on the same set of axis over the interval \([-3, \pi]\).
   (c) Show that the coefficients of the Lagrange polynomials that you found are the columns of the inverse of the Vandermonde matrix for the same polynomial interpolation problem.

2. (Interpolation error) The interpolation points \( x_j, \ j = 0, 1, \ldots n \) influence interpolation error through the polynomial
   \[
   w(x) = \prod_{i=0}^{n} (x - x_i).
   \]
   Suppose we are interpolating the function \( f(x) \) over the interval \([-1, 1]\) using a quadratic interpolant.
   (a) If \( x_0 = -1, \ x_1 = 0, \) and \( x_2 = 1, \) determine the maximum value of the expression \(|(x - x_0)(x - x_1)(x - x_2)|\) for \(-1 \leq x \leq 1\).
   (b) If \( x_0 = -\sqrt{3}/2, \ x_1 = 0, \) and \( x_2 = \sqrt{3}/2, \) determine the maximum value of the expression \(|(x - x_0)(x - x_1)(x - x_2)|\) for \(-1 \leq x \leq 1\). How does this compare to the maximum found in part (2a)?
   (c) Select three random numbers from the interval \([-1, 1]\) to serve as interpolation points \( x_0, \ x_1, \) and \( x_2. \) Determine the maximum value of the expression \(|(x - x_0)(x - x_1)(x - x_2)|\) for \(-1 \leq x \leq 1\) and compare the the maxima found in parts (2a) and (2b). Hint: In Matlab, you can use
   \[
   \text{xdata} = \text{sort}(-1 + 2*\text{rand}(1,3));
   \]
   to get three random numbers in \([-1, 1]\).
   (d) On the same axis, plot the polynomial \( w(x) \) for each of the three sets of interpolating points above. For part (2c), you may plot more than one polynomial.

3. (Newton form of the interpolating polynomial) Construct the divided difference table for the following data set, and then write out the Newton form of the interpolating polynomial.

<table>
<thead>
<tr>
<th>x</th>
<th>-1</th>
<th>0</th>
<th>1</th>
<th>2</th>
</tr>
</thead>
<tbody>
<tr>
<td>y</td>
<td>3</td>
<td>-1</td>
<td>-3</td>
<td>1</td>
</tr>
</tbody>
</table>

Note: We will probably not over this in class, so you will have to read about the Newton form of the interpolating polynomial on your own.
4. (Weierstrass Approximation Theorem) In Figure 1 is a plot of the $\infty$-norm error associated with polynomial interpolation of the function

$$f(x) = \frac{1}{1 + 25x^2}$$

by a degree $n$ polynomial on $n + 1$ equally spaced points on the interval $[-1, 1]$. Note: The divergence of the error is not due to solving the ill-conditioned Vandermonde system.

(a) On the same axis, plot $f(x)$ and the interpolating polynomials for 10, 20 and 30 equally spaced points on the interval $[-1, 1]$. Clearly identify the exact function, and the three interpolating polynomials. You should see evidence of the “Runge phenomena”, the large oscillations near the endpoints of the interval. You may use the Matlab `polyfit` and `polyval` commands. You may also implement the barycentric interpolation formula, although it is not necessary for $n \lesssim 40$ for this function.

(b) Why does the apparent divergence of the error for polynomial interpolation for this function not contradict the Weierstrass Approximation Theorem?

(c) Plot an interpolating polynomial for $f(x)$ at 10, 20 and 30 Chebyshev nodes in the interval $[-1, 1]$. What do you notice?

(d) (Math 565 only) Construct a plot similar to Figure 1 but show the errors associated with polynomial interpolation at Chebyshev interpolation points.

5. (One more question...