Linear systems of equations occur in almost every area of the applied science, engineering, and mathematics.

Hence, numerical linear algebra is one of the pillars of computational mathematics.
A linear system of $m$ equations and $n$ unknowns can be expressed in the following general form:

\[
a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \cdots + a_{1n}x_n = b_1, \\
a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \cdots + a_{2n}x_n = b_2, \\
a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + \cdots + a_{3n}x_n = b_3, \\
\vdots \quad \vdots \quad \vdots \quad \ddots \quad \vdots \quad \vdots \\
a_{m1}x_1 + a_{m2}x_2 + a_{m3}x_3 + \cdots + a_{mn}x_n = b_m.
\] (1)

Here $a_{ij}$ are the coefficients of the systems, $b_i$ are the right hand sides (RHS), and $x_j$ are the unknown values that must be determined. $a_{ij}$ and $b_i$ will be given by the problem.
Linear systems can be classified into the following three types:

1. **Square linear system**: If the number of equations equals the number of unknowns (i.e. $m = n$).

2. **Overdetermined system**: If the number of equations is greater than the number of unknowns (i.e. $m > n$).

3. **Underdetermined system**: If the number of equations is less than the number of unknowns (i.e. $m < n$).
Matrices and vectors

A convenient notation to describe a linear system of equations is in terms of matrices and vectors.
Matrices

A matrix is just a table of numbers containing $m$ rows and $n$ columns and can be expressed as:

$$A = \begin{bmatrix}
a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\
a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\
a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn}
\end{bmatrix}.$$ 

- We typically use capital letters to denote matrices.
- We write $A \in \mathbb{R}^{m \times n}$ to denote a matrix with $m$ rows and $n$ columns.
- A common shorthand notation for a matrix is $A = \{a_{ij}\}$, where the values for $i$ and $j$ are understood from the problem.
Matrices

- A matrix is just a table of numbers containing \( m \) rows and \( n \) columns and can be expressed as:

\[
A = \begin{bmatrix}
   a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\
   a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\
   a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\
   \vdots & \vdots & \vdots & \ddots & \vdots \\
   a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn}
\end{bmatrix}.
\]

- We typically use capital letters to denote matrices.
- We write \( A \in \mathbb{R}^{m \times n} \) to denote a matrix with \( m \) rows and \( n \) columns.
- A common shorthand notation for a matrix is \( A = \{a_{ij}\} \), where the values for \( i \) and \( j \) are understood from the problem.
A matrix is just a table of numbers containing $m$ rows and $n$ columns and can be expressed as:

$$A = \begin{bmatrix}
a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\
a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\
a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn}
\end{bmatrix}.$$ 

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We write $A \in \mathbb{R}^{m \times n}$ to denote a matrix with $m$ rows and $n$ columns.

A common shorthand notation for a matrix is $A = \{a_{ij}\}$, where the values for $i$ and $j$ are understood from the problem.
Matrices

A matrix is just a table of numbers containing $m$ rows and $n$ columns and can be expressed as:

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{bmatrix}.$$ 

- We typically use capital letters to denote matrices.
- We write $A \in \mathbb{R}^{m \times n}$ to denote a matrix with $m$ rows and $n$ columns.
- A common shorthand notation for a matrix is $A = \{a_{ij}\}$, where the values for $i$ and $j$ are understood from the problem.
Vectors

If the matrix only has one row or column then it is called a vector.

- A column vector with \( n \) entries can be expressed as

\[
\begin{bmatrix}
    x_1 \\
    x_2 \\
    x_3 \\
    \vdots \\
    x_n
\end{bmatrix}.
\]

- A row vector and can be expressed as

\[
\begin{bmatrix}
    x_1 & x_2 & x_3 & \cdots & x_n
\end{bmatrix}.
\]

- We typically use bold lower-case letters to denote vectors.
- A column vector with \( n \) real entries is denoted by \( \mathbf{x} \in \mathbb{R}^n \), while a row vector is denoted by \( \mathbf{x} \in \mathbb{R}^{1 \times n} \).
If the matrix only has one row or column then it is called a vector.

- A **column vector** with \( n \) entries can be expressed as

\[
x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix}.
\]

- A **row vector** can be expressed as

\[
x = [x_1 \ x_2 \ x_3 \ \cdots \ x_n].
\]

We typically use bold lower-case letters to denote vectors.

- A column vector with \( n \) real entries is denoted by \( x \in \mathbb{R}^n \), while a row vector is denoted by \( x \in \mathbb{R}^{1 \times n} \).
If the matrix only has one row or column then it is called a vector.

- A **column vector** with \( n \) entries can be expressed as

\[
\mathbf{x} = \begin{bmatrix}
    x_1 \\
    x_2 \\
    x_3 \\
    \vdots \\
    x_n
\end{bmatrix}.
\]

- A **row vector** and can be expressed as

\[
\mathbf{x} = [x_1 \ x_2 \ x_3 \ \cdots \ x_n].
\]

We typically use bold lower-case letters to denote vectors.

- A column vector with \( n \) real entries is denoted by \( \mathbf{x} \in \mathbb{R}^n \), while a row vector is denoted by \( \mathbf{x} \in \mathbb{R}^{1 \times n} \).
If the matrix only has one row or column then it is called a vector.

- A column vector with \(n\) entries can be expressed as
  
  \[
  \mathbf{x} = \begin{bmatrix}
  x_1 \\
  x_2 \\
  x_3 \\
  \vdots \\
  x_n
  \end{bmatrix}.
  \]

- A row vector and can be expressed as
  
  \[
  \mathbf{x} = \begin{bmatrix}
  x_1 & x_2 & x_3 & \cdots & x_n
  \end{bmatrix}.
  \]

- We typically use bold lower-case letters to denote vectors.
  - A column vector with \(n\) real entries is denoted by \(\mathbf{x} \in \mathbb{R}^n\), while a row vector is denoted by \(\mathbf{x} \in \mathbb{R}^{1 \times n}\).
Vectors

If the matrix only has one row or column then it is called a vector.

- A **column vector** with $n$ entries can be expressed as

  \[ \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix}. \]

- A **row vector** and can be expressed as

  \[ \mathbf{x} = [x_1 \ x_2 \ x_3 \ \cdots \ x_n]. \]

We typically use bold lower-case letters to denote vectors.

A column vector with $n$ real entries is denoted by $\mathbf{x} \in \mathbb{R}^n$, while a row vector is denoted by $\mathbf{x} \in \mathbb{R}^{1 \times n}$. 
Matrix & vector operations
Matrix & vector operations
Matrix & vector operations: Transpose

Let $A \in \mathbb{R}^{m \times n}$ with entries

$$A = \begin{bmatrix}
  a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\
  a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\
  a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\
  \vdots & \vdots & \vdots & \ddots & \vdots \\
  a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn}
\end{bmatrix},$$

then the \textbf{transpose} of $A$ switches the columns of $A$ with the rows, i.e.

$$A^T = \begin{bmatrix}
  a_{11} & a_{21} & a_{31} & \cdots & a_{m1} \\
  a_{12} & a_{22} & a_{32} & \cdots & a_{m2} \\
  a_{13} & a_{23} & a_{33} & \cdots & a_{m3} \\
  \vdots & \vdots & \vdots & \ddots & \vdots \\
  a_{1n} & a_{2n} & a_{3n} & \cdots & a_{mn}
\end{bmatrix}.$$

Note that $A^T \in \mathbb{R}^{n \times m}$ and that $(A^T)^T = A$. 
The transpose can also be applied to vectors. In this case if $\mathbf{x}$ is a (column) vector then $\mathbf{x}^T$ is a row vector:

$$
\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix} \quad \text{then} \quad \mathbf{x}^T = [x_1 \ x_2 \ x_3 \ \cdots \ x_n].
$$

Similarly if $\mathbf{x}$ is row vector then $\mathbf{x}^T$ is a column vector.
Let $A \in \mathbb{R}^{m \times n}$ and $B \in \mathbb{R}^{m \times n}$ then the sum of $A$ and $B$ is given by

$$A + B = \left\{ a_{ij} + b_{ij} \right\}.$$ 

This is just the sum of the corresponding entries of the elements of $A$ and $B$.

For this sum to make sense $A$ and $B$ must be the same size.
Let $\alpha$ be a real number and $A \in \mathbb{R}^{m \times n}$ then the product of $\alpha$ and $A$ is given by

$$\alpha A = \begin{bmatrix} \alpha a_{ij} \end{bmatrix}.$$ 

Note that this is just $\alpha$ times each entry of $A$. 
There are two types of vector-vector products that arise quite frequently. These can be derived from the definition for matrix-matrix products (discussed later), but it is worth stating them separately.

- Let $x, y \in \mathbb{R}^n$ then the **inner product** or **dot product** of $x$ and $y$ is

$$
\mathbf{x}^T \mathbf{y} = \begin{bmatrix} x_1 & x_2 & \cdots & x_n \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} = x_1 y_1 + x_2 y_2 + \cdots + x_n y_n = \sum_{j=1}^{n} x_j y_j.
$$

Note that the inner product is a single number. The inner product is sometimes denoted by $\mathbf{x} \cdot \mathbf{y}$. 
There are two types of vector-vector products that arise quite frequently. These can be derived from the definition for matrix-matrix products (discussed later), but it is worth stating them separately.

- Let $\mathbf{x} \in \mathbb{R}^m$ and $\mathbf{y} \in \mathbb{R}^n$ then the **outer product** of $\mathbf{x}$ with $\mathbf{y}$ is

$$\mathbf{xy}^T = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{bmatrix} \begin{bmatrix} y_1 & y_2 & \cdots & y_n \end{bmatrix} = \begin{bmatrix} x_1y_1 & x_1y_2 & \cdots & x_1y_n \\ x_2y_1 & x_2y_2 & \cdots & x_2y_n \\ \vdots & \vdots & \ddots & \vdots \\ x=my_1 & x=my_2 & \cdots & x=my_n \end{bmatrix}$$

Note that the outer product is a matrix of size $m$-by-$m$. 
Let $A \in \mathbb{R}^{m \times n}$ and $x \in \mathbb{R}^n$ then the product of $A$ and $x$ is given by

$$\begin{bmatrix} a_{11} \\ a_{21} \\ a_{31} \\ \vdots \\ a_{m1} \end{bmatrix} + x_2 \begin{bmatrix} a_{12} \\ a_{22} \\ a_{32} \\ \vdots \\ a_{m2} \end{bmatrix} + x_3 \begin{bmatrix} a_{13} \\ a_{23} \\ a_{33} \\ \vdots \\ a_{m3} \end{bmatrix} + \ldots + x_n \begin{bmatrix} a_{1n} \\ a_{2n} \\ a_{3n} \\ \vdots \\ a_{mn} \end{bmatrix}$$

(2)

Thus, the product $Ax$ is a linear combination of the columns of $A$. 

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Linear Algebra Basics

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Important observations regarding the matrix-vector product $A\mathbf{x}$:

- The only way for this product to make sense is if $A$ has the same number of columns as $\mathbf{x}$ does rows.
- $A\mathbf{x} \in \mathbb{R}^m$, i.e. the product is a column vector containing $m$ entries.
- If we let $\mathbf{b} = A\mathbf{x}$ then we can alternatively express the $i$th entry of $\mathbf{b}$ as

$$b_i = \sum_{j=1}^{n} a_{ij}x_j, \quad i = 1, \ldots, m.$$  

This illustrates that $b_i$ is just the inner product of the $i$th row of $A$ with the vector $\mathbf{x}$.

- In general, computing $A\mathbf{x}$ using the above formulas requires $mn$ multiplications and $m(n - 1)$ additions.
Let $A \in \mathbb{R}^{m \times n}$ and $B \in \mathbb{R}^{n \times p}$, and let $B$ have columns

$$B = \begin{bmatrix} b_1 & b_2 & \cdots & b_n \end{bmatrix}.$$ 

The matrix-matrix product $C = AB$ is given as

$$C = \begin{bmatrix} Ab_1 & Ab_2 & \cdots & Ab_n \end{bmatrix}.$$ 

This shows the $k$th column of the product $AB$ is a linear combination of the columns of $A$ with the coefficients in the linear combinations being determined by entries in the $k$th column of $B$. 
Important observations regarding the matrix-matrix product $AB$, $A \in \mathbb{R}^{m \times n}$, $B \in \mathbb{R}^{n \times p}$:

- Number rows of $A$ must equal number columns $B$.
- $AB \in \mathbb{R}^{m \times p}$, i.e. the product is a matrix containing $m$ rows and $p$ columns.
- In general, $AB \neq BA$, i.e. the product does not commute.
Important observations regarding the matrix-matrix product $AB$, $A \in \mathbb{R}^{m \times n}$, $B \in \mathbb{R}^{n \times p}$:

- We can express each entry of $C$ as

$$c_{ik} = \sum_{j=1}^{n} a_{ij} b_{jk}, \quad i = 1, \ldots, m, \quad k = 1, \ldots, p.$$  

So $c_{ik}$ is just the inner product of the $i$th row of $A$ with the $k$th column of $B$.

- Computing $AB$ using the above formulas requires $mnp$ multiplications and $m(n - 1)p$ additions.

- The transpose of the product $AB$ satisfies: $(AB)^T = B^T A^T$. 
Recall that we can express a linear system of equations with $m$ equations and $n$ unknowns as

$$
\begin{align*}
    a_{11} x_1 & + a_{12} x_2 + a_{13} x_3 + \cdots + a_{1n} x_n = b_1, \\
    a_{21} x_1 & + a_{22} x_2 + a_{23} x_3 + \cdots + a_{2n} x_n = b_2, \\
    a_{31} x_1 & + a_{32} x_2 + a_{33} x_3 + \cdots + a_{3n} x_n = b_3, \\
    \vdots & \quad \vdots \quad \vdots \quad \ddots \quad \vdots \quad \vdots \quad \vdots \\
    a_{m1} x_1 & + a_{m2} x_2 + a_{m3} x_3 + \cdots + a_{mn} x_n = b_m.
\end{align*}
$$

We can express this linear system in matrix-vector notation using the previous definitions.
Linear systems in matrix-vector notation

Let \( \mathbf{x} \in \mathbb{R}^n \), \( \mathbf{b} \in \mathbb{R}^m \), and \( A \in \mathbb{R}^{m \times n} \), then the linear system is given as \( A \mathbf{x} = \mathbf{b} \), or

\[
\begin{bmatrix}
  a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\
  a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\
  a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\
  \vdots & \vdots & \vdots & \ddots & \vdots \\
  a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn}
\end{bmatrix}
\begin{bmatrix}
  x_1 \\
  x_2 \\
  x_3 \\
  \vdots \\
  x_n
\end{bmatrix}
=
\begin{bmatrix}
  b_1 \\
  b_2 \\
  b_3 \\
  \vdots \\
  b_m
\end{bmatrix}.
\]
Recall that $Ax$ is a linear combination of the columns of $A$:

$Ax = x_1 \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix} + x_2 \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix} + x_3 \begin{bmatrix} a_{13} \\ a_{23} \\ \vdots \\ a_{m3} \end{bmatrix} + \ldots + x_n \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix}.$

Thus, the only way there will be a solution to $Ax = b$ is if $b$ can be written as a linear combination of the columns of $A$. 
Linear systems: solvability

There are three possibilities for the linear system $Ax = b$:

1. There are an **infinite number of solutions** that satisfy $Ax = b$.  
   An infinite number of ways to linearly combine the columns of $A$ to equal $b$.

2. There is **one unique solution** to the linear system.  
   Only one way to linearly combine the columns of $A$ to equal $b$.

3. There is **no solution** to the linear system.  
   There is no way to linearly combine the columns of $A$ to equal $b$. 

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A diagonal matrix is an $n$-by-$n$ square matrix with zeros on in every entry except possibly the main diagonal:

$$D = \begin{bmatrix}
d_1 & 0 & 0 & \ldots & 0 \\
0 & d_2 & 0 & \ldots & 0 \\
0 & 0 & d_3 & \ldots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \ldots & d_n
\end{bmatrix},$$

where $d_j, j = 1, \ldots, n$ are some real numbers.
The identity matrix is a diagonal matrix with every diagonal entry equal to 1:

\[
I = \begin{bmatrix}
1 & 0 & 0 & \cdots & 0 \\
0 & 1 & 0 & \cdots & 0 \\
0 & 0 & 1 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & 1
\end{bmatrix}
\]

It has the property that for any matrix \( A \in \mathbb{R}^{n \times n} \), \( IA = AI = A \).
A matrix $L \in \mathbb{R}^{m\times n}$ is lower triangular if all the entries above its main diagonal are zero. Square $n$-by-$n$ lower triangular matrices take the form

$$L = \begin{bmatrix}
\ell_{11} & 0 & 0 & \cdots & 0 \\
\ell_{21} & \ell_{22} & 0 & \cdots & 0 \\
\ell_{31} & \ell_{32} & \ell_{33} & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
\ell_{n1} & \ell_{n2} & \ell_{n3} & \cdots & \ell_{nn}
\end{bmatrix},$$

where $\ell_{i,j}$, $i = 1, \ldots, n$, $j = i, \ldots, n$ are some real numbers.
A matrix $U \in \mathbb{R}^{m \times n}$ is upper triangular if all the entries below its main diagonal are zero. Square $n$-by-$n$ upper triangular matrices take the form

$$U = \begin{bmatrix}
    u_{11} & u_{12} & u_{13} & \cdots & u_{1n} \\
    0 & u_{22} & u_{23} & \cdots & u_{2n} \\
    0 & 0 & u_{33} & \cdots & u_{3n} \\
    \vdots & \vdots & \vdots & \ddots & \vdots \\
    0 & 0 & 0 & \cdots & u_{nn}
\end{bmatrix},$$

where $u_{i,j}$, $i = 1, \ldots, n$, $j = i, \ldots, n$ are some real numbers.
A matrix $A$ is symmetric if $A = A^T$. Note that only square matrices can be symmetric.
Inverse of a matrix

Let $A$ be an $n$-by-$n$ square matrix (i.e. $A \in \mathbb{R}^{n \times n}$). If there exists a square matrix $B \in \mathbb{R}^{n \times n}$ such that

$$BA = AB = I,$$

where $I$ is the $n$-by-$n$ identity matrix, then $B$ is called the inverse of $A$.

- The inverse of $A$ is denoted by $A^{-1}$.
- If $A^{-1}$ exists then $A$ is called nonsingular, otherwise it is singular.
If $A$ is a square, nonsingular matrix, then the solution to the linear system $Ax = b$ is given formally as

$$x = A^{-1}b.$$ 

**Important:** When solving a linear system, one should never first compute $A^{-1}$ and then compute the product $A^{-1}b$. There are much better ways to solve the system (for example using Gaussian *elimination* when $n$ is not too large).
Suppose $A$ is nonsingular then the following statements are true

- $A^{-1}$ is unique
- $A^{-1}$ is nonsingular and its inverse is $A$
- $A^T$ is nonsingular
- If $B \in \mathbb{R}^{n \times n}$ is nonsingular then $AB$ is nonsingular and $(AB)^{-1} = B^{-1}A^{-1}$
- The linear system $Ax = b$ has a unique solution.
- A vector norm is a scalar quantity that reflects the “size” of a vector $x$.
- The norm of a vector $x$ is denoted as $\|x\|$.
- There are many ways to define the size of a vector. If $x \in \mathbb{R}^n$, the three most popular are

  - one-norm: $\|x\|_1 = \sum_{k=1}^{n} |x_k|$
  - two-norm: $\|x\|_2 = \sqrt{\sum_{k=1}^{n} |x_k|^2}$
  - $\infty$-norm: $\|x\|_{\infty} = \max_{1 \leq k \leq n} |x_k|$
However, a vector norm is defined, it must satisfy the following three properties to be called a norm:

1. $\|x\| \geq 0$ and $\|x\| = 0$ if and only if $x = 0$ (i.e. $x$ contains all zeros as its entries).

2. $\|\alpha x\| = |\alpha| \|x\|$, for any constant $\alpha$.

3. $\|x + y\| \leq \|x\| + \|y\|$, where $y \in \mathbb{R}^n$. This is called the triangle inequality.
Unit vectors

- A vector $\mathbf{x}$ is called a **unit vector** if its norm is one, i.e. $\|\mathbf{x}\| = 1$.
- Unit vectors will be different depending on the norm applied.
- Below are several unit vectors in the one, two, and $\infty$ norms for $\mathbf{x} \in \mathbb{R}^2$.

(a) One-norm

(b) Two-norm

(c) $\infty$-norm
A matrix norm is a scalar quantity that reflects the “size” of a matrix $A \in \mathbb{R}^{m \times n}$.

The norm of $A$ is denoted as $\|A\|$. 

Any matrix norm must satisfy the following four properties:

1. $\|A\| \geq 0$ and $\|A\| = 0$ if and only if $A = 0$ (i.e. $A$ contains all zeros as its entries).
2. $\|\alpha A\| = |\alpha|\|A\|$, for any constant $\alpha$.
3. $\|A + B\| \leq \|A\| + \|B\|$, where $B \in \mathbb{R}^{m \times n}$.
4. $\|AB\| \leq \|A\|\|B\|$, where $B \in \mathbb{R}^{n \times p}$. This is called the submultiplicative inequality.
Matrix norms

Each vector norm induces a matrix norm according to the following definition:

$$\|A\|_p = \max_{\|x\|_p \neq 0} \frac{\|Ax\|_p}{\|x\|_p} = \max_{\|x\|_p = 1} \|Ax\|_p,$$

where $x \in \mathbb{R}^n$ and $p = 1, 2, \ldots$.

Induced norms describe how the matrix stretches unit vectors with respect to that norm.
Induced matrix norms

Two popular and easy to define induced matrix norms are

One-norm:  \[ \|A\|_1 = \max_{1 \leq k \leq n} \sum_{j=1}^{m} |a_{jk}|, \]

\[ \|A\|_{\infty} = \max_{1 \leq j \leq m} \sum_{k=1}^{n} |a_{jk}|. \]

- The one-norm corresponds to the maximum of the one norm of every column.
- The \( \infty \)-norm corresponds to the maximum of the one norm of every row.

The two-norm of \( A \) is defined as the *largest eigenvalue* of the matrix \( A^T A \). This is computationally expensive to compute.
Non-induced matrix norms

The most popular matrix norm that is not an induced norm is the Frobenius norm:

$$\|A\|_F = \sqrt{\sum_{j=1}^{m} \sum_{k=1}^{n} |a_{jk}|^2}.$$
Important results on matrix norms

The following are some useful inequalities involving matrix norms. Here $A \in \mathbb{R}^{m \times n}$:

- $\|Ax\| \leq \|A\| \|x\|$
- $\frac{1}{\sqrt{m}} \|A\|_1 \leq \|A\|_2 \leq \sqrt{n} \|A\|_1$
- $\frac{1}{\sqrt{n}} \|A\|_\infty \leq \|A\|_2 \leq \sqrt{m} \|A\|_\infty$
- $\|A\|_2 \leq \sqrt{\|A\|_1 \|A\|_\infty}$