Typical linear system of equations:

\[
\begin{align*}
5x_1 - x_2 + 2x_3 &= 7 \\
-2x_1 + 6x_2 + 9x_3 &= 0 \\
-7x_1 + 5x_2 - 3x_3 &= 5
\end{align*}
\]

The variables \(x_1, x_2,\) and \(x_3\) only appear as linear terms (no powers or products).

This is a square linear system, since we have the same number of equations as variables.
Where do linear systems come from?

- Fitting curves to data
- Polynomial approximation to functions
- Computational fluid dynamics
- Network flow
- Computer graphics
- Difference equations
- Differential equations
- Dynamical systems theory
Typical linear system

How does Matlab solve linear systems such as:

\[
\begin{align*}
5x_1 - x_2 + 2x_3 &= 7 \\
-2x_1 + 6x_2 + 9x_3 &= 0 \\
-7x_1 + 5x_2 - 3x_3 &= 5
\end{align*}
\]

- Does such a system always have a solution?
- Can such a system be solved efficiently for millions of equations?
- What does Matlab do if we have more equations than unknowns? More unknowns than equations?
Solving linear systems

We are already familiar with at least one type of linear system
\[
3x_1 + 5x_2 = 3 \\
2x_1 - 4x_2 = 1
\]

The solution is the intersection of the two lines represented by each equation. This solution is a point \((x_1, x_2)\) that satisfies both equations simultaneously.

We could also view the solution as providing the correct linear combination of vectors \((3, 2)\) and \((5, -4)\) that give us \((3, 1)\).

\[
x_1 \begin{bmatrix} 3 \\ 2 \end{bmatrix} + x_2 \begin{bmatrix} 5 \\ -4 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}
\]
Linear algebra - a 2x2 system

We can row-reduce an augmented matrix to find the solution:

\[
\begin{bmatrix}
3 & 5 & | & 3 \\
2 & -4 & | & 1
\end{bmatrix}
\rightarrow
\begin{bmatrix}
3 & 5 & | & 3 \\
0 & -\frac{22}{3} & | & -1
\end{bmatrix}
\leftarrow \text{eqn 2) } - \left( \frac{2}{3} \right) \text{ (eqn 1)}
\]

Use an elementary row operation to produce a “0” in the lower left corner.

Use back-substitution to solve first for \( x_2 \) and then for \( x_1 \).

\[
x_2 = \left( \frac{-3}{22} \right) (-1) = \frac{3}{22}
\]

\[
x_1 = \frac{1}{3} (3 - 5x_2) = \frac{1}{3} \left( 3 - 5 \left( \frac{-3}{22} \right) \right) = \frac{17}{22}
\]

Solution: \( x_1 = \frac{17}{22}, \quad x_2 = \frac{3}{22} \)
Linear algebra - a 2x2 system

Solution as the intersection of two lines

Solution as linear combination of vectors